

# THE EQUIVALENCE OF HEEGAARD FLOER HOMOLOGY AND EMBEDDED CONTACT HOMOLOGY VIA OPEN BOOK DECOMPOSITIONS I

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**ABSTRACT.** Given an open book decomposition  $(S, h)$  adapted to a closed, oriented 3-manifold  $M$ , we define a chain map  $\Phi$  from a certain Heegaard Floer chain complex associated to  $(S, h)$  to a certain embedded contact homology chain complex associated to  $(S, h)$ , as defined in [CGH1], and prove that it induces an isomorphism on the level of homology. This implies the isomorphism between the hat version of Heegaard Floer homology of  $-M$  and the hat version of embedded contact homology of  $M$ .

## CONTENTS

1. Introduction and main results	3
1.1. Main result	3
1.2. Outline of proof.	4
1.3. Organization of the paper.	5
2. Adapting $ECH$ to an open book decomposition	5
2.1. The first return map	5
2.2. $ECH(N, \partial N, \alpha)$ and $\widehat{ECH}(N, \partial N, \alpha)$	6
2.3. Splitting of $ECH$ according to homology classes	7
2.4. Twisted coefficients in $ECH$	8
2.5. Elimination of elliptic orbits	9
3. Periodic Floer homology	12
3.1. Interpolating between Reeb and stable Hamiltonian vector fields	12
3.2. Definitions	13
3.3. The flux	13
3.4. Compactness	14
3.5. Transversality	15
3.6. The equivalence of certain $ECH$ and $PFH$ groups	16
4. A variation of $\widehat{HF}(M)$ adapted to open book decompositions	17

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*Date:* This version: July 24, 2012.

*2000 Mathematics Subject Classification.* Primary 57M50; Secondary 53D10, 53D40.

*Key words and phrases.* contact structure, Reeb dynamics, embedded contact homology, Heegaard Floer homology, open book decompositions.

VC supported by the Institut Universitaire de France, ANR Symplexe, ANR Floer Power, and ERC Geodycon. PG supported by ANR Floer Power and ANR TCGD. KH supported by NSF Grants DMS-0805352 and DMS-1105432.

4.1.	Heegaard data	17
4.2.	Almost complex structures	17
4.3.	Holomorphic curves and moduli spaces	18
4.4.	The Fredholm index	19
4.5.	The ECH-type index	21
4.6.	Compactness	26
4.7.	Transversality	27
4.8.	Definition of the Heegaard Floer homology groups	27
4.9.	Restricting the complex to a page	28
4.10.	$\text{Spin}^c$ -structures	31
4.11.	Twisted coefficients in Heegaard Floer homology	33
5.	Moduli spaces of multisections	34
5.1.	Symplectic cobordisms	34
5.2.	Lagrangian boundary conditions	37
5.3.	Almost complex structures and moduli spaces for $W$ , $\overline{W}$ , $W'$ , and $\overline{W}'$	38
5.4.	Almost complex structures and moduli spaces for $W_+$ , $\overline{W}_+$ , $W_-$ , and $\overline{W}_-$	41
5.5.	The Fredholm index	45
5.6.	The ECH index	50
5.7.	Holomorphic curves with ends at $z_\infty$	52
5.8.	Transversality	63
6.	The chain map from $\widehat{HF}$ to $PFH$	66
6.1.	Compactness for $W_+$	66
6.2.	Definition of $\Phi$	71
6.3.	$\text{Spin}^c$ -structures	73
6.4.	Twisted coefficients	74
6.5.	Gluing	74
6.6.	The variant $\tilde{\Phi}$	79
7.	The chain map from $PFH$ to $\widehat{HF}$	81
7.1.	Definition of $\Psi$	81
7.2.	Outline of proof of Theorem 7.1.3	83
7.3.	SFT compactness	85
7.4.	Novelty of the $\overline{W}_-$ case	86
7.5.	Intersection numbers	87
7.6.	Some restrictions on $\overline{u}_\infty$	89
7.7.	Compactness theorem	94
7.8.	Asymptotic eigenfunctions	96
7.9.	The rescaled function	100
7.10.	Involution lemmas	110
7.11.	Elimination of some cases	113
7.12.	Proof of Lemma 7.2.3	115
7.13.	Gluing	116
	References	125

## 1. INTRODUCTION AND MAIN RESULTS

This is the second in a series of papers devoted to proving the equivalence of Heegaard Floer homology and embedded contact homology. The goal of this paper and its sequel [CGH2] is to establish an isomorphism between the hat versions of the Heegaard Floer homology and embedded contact homology (abbreviated ECH) groups associated to a closed, oriented 3-manifold  $M$ . The results of this paper were announced in [CGH0]. The isomorphism between the plus version of Heegaard Floer homology and the usual version of ECH will be given in [CGH3].

Heegaard Floer homology, defined by Ozsváth-Szabó [OSz1, OSz2], *a priori* depends on the choice of a Heegaard surface  $\Sigma$  for  $M$ , a basepoint  $z \in \Sigma$ , totally real tori  $\mathbb{T}_\alpha, \mathbb{T}_\beta$  in  $Sym^g(\Sigma)$ , and an almost complex structure on  $Sym^g(\Sigma)$ . However, it was shown to be independent of those choices, i.e., is a topological invariant of  $M$ . On the other hand, ECH, defined by Hutchings [Hu1, Hu2, HT1, HT2] *a priori* depends on the choice of a contact form  $\alpha$  on  $M$  and an adapted almost complex structure  $J$  on the symplectization  $\mathbb{R} \times M$ . There is currently no direct proof of the fact that the ECH groups are topological invariants of  $M$  (or even invariants of the contact structure  $\ker \alpha$ , for that matter); the only known proof is due to Taubes [T1, T2], and is a consequence of the equivalence between Seiberg-Witten Floer cohomology and ECH.

*In this paper we will use  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$  coefficients (or coefficients in a module  $\Lambda$  over  $\mathbb{F}[H_2(M; \mathbb{Z})]$ ) for both Heegaard Floer homology and ECH.*

A natural setting for relating Heegaard Floer homology and ECH is that of *open book decompositions*. In the foundational work [Gi2], Giroux proved the equivalence of contact structures up to isomorphism and (abstract) open book decompositions modulo stabilization. Let  $(S, h)$  be a pair consisting of a compact oriented surface  $S$  with nonempty boundary (sometimes called a *bordered surface*) and a diffeomorphism  $h : S \xrightarrow{\sim} S$  which restricts to the identity on  $\partial S$ . Let  $K \subset M$  be a link. Then  $M$  admits an *open book decomposition*  $(S, h)$  with binding  $K$  if there is a homeomorphism

$$((S \times [0, 1]) / \sim, (\partial S \times [0, 1]) / \sim) \simeq (M, K).$$

Here the equivalence relation  $\sim$  is generated by  $(x, 1) \sim (h(x), 0)$  for  $x \in S$  and  $(y, t) \sim (y, t')$  for  $y \in \partial S$  and  $t, t' \in [0, 1]$ . Let  $\xi_{(S, h)}$  be the contact structure that corresponds to  $(S, h)$  under the Giroux correspondence. *In this paper we assume that  $\partial S$  is connected, unless stated otherwise.*

**1.1. Main result.** The main result of this paper and its sequel [CGH2] is the following:

**Theorem 1.1.1.** *There is an isomorphism*

$$\Phi_0 : \widehat{HF}(-M, \mathfrak{s}_{\xi_{(S, h)}} + PD(A)) \xrightarrow{\sim} \widehat{ECH}(M, \xi_{(S, h)}, A),$$

where  $A \in H_1(M; \mathbb{Z})$ , defined via an open book decomposition  $(S, h)$  of  $M$ . Moreover,  $\Phi_0$  sends the Heegaard Floer contact invariant for  $\xi_{(S, h)}$  to the ECH contact invariant for  $\xi_{(S, h)}$ .

There is a similar map for the so-called twisted coefficients. Let  $\Lambda$  be any module over the group ring  $\mathbb{F}[H_2(M; \mathbb{Z})]$ . We denote the versions of Heegaard Floer homology and ECH with twisted coefficients in  $\Lambda$  by  $\widehat{HF}(-M, \mathfrak{s}; \Lambda)$  and  $\widehat{ECH}(M, \xi, A; \Lambda)$ . Twisted coefficients are defined in [OSz2, Section 8] for Heegaard Floer homology and in [HS2, Section 11] for ECH.

**Theorem 1.1.2.** *There is an  $\mathbb{F}[H_2(M; \mathbb{Z})]$ -module isomorphism*

$$\Phi_0 : \widehat{HF}(-M, \mathfrak{s}_{\xi(S,h)} + PD(A); \Lambda) \xrightarrow{\sim} \widehat{ECH}(M, \xi_{(S,h)}, A; \Lambda),$$

where  $A \in H_1(M; \mathbb{Z})$ , defined via an open book decomposition  $(S, h)$  of  $M$ . Moreover,  $\Phi_0$  sends the Heegaard Floer contact invariant for  $\xi_{(S,h)}$  to the ECH contact invariant for  $\xi_{(S,h)}$ .

On the other hand, Taubes [T2] has proven that Seiberg-Witten Floer cohomology and ECH are isomorphic. Let  $\widetilde{HM}(M)$  be the homology of the mapping cone of  $U_{\dagger} : \check{C}(M) \rightarrow \check{C}(M)$ , where  $\check{C}(M)$  is the chain complex for  $\widetilde{HM}(M)$ . Combining Taubes' theorem with Theorem 1.1.1, we obtain the following ‘‘corollary’’:

**Corollary 1.1.3.**  $\widehat{HF}(M, \mathfrak{s}) \simeq \widetilde{HM}(M, \mathfrak{s})$  for any  $\mathfrak{s} \in \text{Spin}^c(M)$ .

An alternate proof of the HF=ECH correspondence has recently been given by Kutluhan-Lee-Taubes [KLT1]–[KLT5].

**1.2. Outline of proof.** The isomorphism  $\Phi_0$  is defined via an intermediary  $N = N_{(S,h)}$ , called the *suspension* of  $(S, h)$ , and which is given as follows:

$$N = S \times [0, 1] / \sim, \quad (x, 1) \sim (h(x), 0).$$

In the first paper of our series [CGH1], we introduced the ECH group  $\widehat{ECH}(N, \partial N)$ , given as a direct limit

$$\widehat{ECH}(N, \partial N) = \lim_{j \rightarrow \infty} ECH_j(N),$$

where  $j$  is the number of intersections of an orbit set with a page  $S \times \{0\}$  of an open book and the direct limit is taken with respect to the maps

$$(\mathfrak{I}_j)_* : ECH_j(N) \rightarrow ECH_{j+1}(N)$$

induced by inclusions. The following was proved in [CGH1]:

**Theorem 1.2.1.**  $\widehat{ECH}(M) \simeq \widehat{ECH}(N, \partial N)$ .

The map  $\Phi_0$  is, roughly speaking, the composition of

$$\Phi_* : \widehat{HF}(-M) \rightarrow ECH_{2g}(N),$$

induced by a symplectic cobordism  $W_+$ , followed by the natural map

$$ECH_{2g}(N) \rightarrow \lim_{j \rightarrow \infty} ECH_j(N),$$

induced by the maps  $(\mathfrak{I}_j)_*$ . Here  $g$  is the genus of  $S$ . We prove that the map  $\Phi_*$  is an isomorphism, and then prove that the maps  $(\mathfrak{I}_j)_*$  are isomorphisms for  $j \geq 2g$ . (Strictly speaking, we need to use a ‘‘perturbed’’ version of  $\lim_{j \rightarrow \infty} ECH_j(N)$ ,

as explained in Section 2.5, and replace the ECH groups by certain periodic Floer homology groups, as explained in Section 3.)

**1.3. Organization of the paper.** References from [CGH2] will be written as “Section II.x” to mean “Section x” of [CGH2], for example.

In Section 2 we recall some results of [CGH1], including the definition of  $\widehat{ECH}(N, \partial N)$ . In Section 3 we replace the ECH chain complexes  $ECC_j(N)$  by the periodic Floer homology chain complexes  $PFC_j(N)$ , which are technically a little easier to use when defining chain maps to and from Heegaard Floer homology. Then in Section 4 we (i) review Lipshitz’ reformulation of Heegaard Floer homology, (ii) restrict the Heegaard Floer chain complex to a page  $S$  as in [HKM1] and obtain the chain group  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$  whose homology is isomorphic to  $\widehat{HF}(-M)$ , and (iii) introduce an ECH-type index  $I_{HF}$  for Heegaard Floer homology. Section 5 is devoted to describing the moduli spaces of multisections which are used in the definitions of the chain maps  $\Phi$  and  $\Psi$  between the Heegaard Floer chain complex  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$  and the periodic Floer homology chain complex  $PFC_{2g}(N)$ . Then in Sections 6 and 7 we show that  $\Phi$  and  $\Psi$  are indeed chain maps. The proof that  $\Psi$  is a chain map is substantially more involved than the proof that  $\Phi$  is a chain map.

The proofs of the chain homotopies between the chain maps  $\Psi \circ \Phi$  and  $id$ , and between the chain maps  $\Phi \circ \Psi$  and  $id$ , are rather involved and occupy almost all (Sections II.2–II.4) of the sequel [CGH2]. The necessary Gromov-Witten type calculations which are used in the proof of the chain homotopy are carried out in Section II.2. Finally, in Section II.5 we prove that the close cousins of the maps  $(\mathfrak{J}_j)_*$  are isomorphisms for  $j \geq 2g$ .

## 2. ADAPTING $ECH$ TO AN OPEN BOOK DECOMPOSITION

In this section we briefly recall the results of [CGH1]. The reader is referred to [CGH1] for a more complete discussion; the notation here is the same as that of [CGH1], unless indicated otherwise.

**2.1. The first return map.** Let  $S$  be a bordered surface. Let  $N(\partial S) \simeq [-\varepsilon, 0] \times \mathbb{R}/\mathbb{Z}$  be a neighborhood of  $\partial S$  with coordinates  $(y, \theta)$  so that  $\partial S = \{y = 0\}$ . (The slight difference with [CGH1] is that, in [CGH1, Section 2.3],  $\partial S = \{y = 1\}$  instead of  $\{y = 0\}$ .) Let  $\text{Diff}(S, \partial S)$  be the group of diffeomorphisms  $h$  of  $S$  which restrict to the identity on  $\partial S$ . Given a diffeomorphism  $h \in \text{Diff}(S, \partial S)$ , we define the suspension

$$N_{(S,h)} = S \times [0, 1] / \sim,$$

where  $(x, 1) \sim (h(x), 0)$ .

The following was essentially proved in [CGH1]:

**Lemma 2.1.1.** *Given  $h \in \text{Diff}(S, \partial S)$ , there exists  $h_0 \in \text{Diff}(S, \partial S)$  in the same connected component as  $h$ , and which satisfies the following:*

- (1) *there exists a contact form  $\alpha$  on  $N_{(S,h)}$  such that the Reeb vector field  $R_\alpha$  of  $\alpha$  is transverse to  $S \times \{t\}$ ,  $t \in [0, 1]$ , and  $h_0$  is the first return map of  $R_\alpha$  on  $S \times \{0\}$ ; and*
- (2) *the diffeomorphism  $h_0$  restricts to  $(y, \theta) \mapsto (y, \theta - y)$  on  $N(\partial S)$ .*

Moreover,  $\alpha = f_t dt + \beta_t$ , where  $f_t$  is a positive function on  $S$  and  $\beta_t$  is a Liouville 1-form on  $S$ .

*Proof.* (1) and the last sentence of the lemma follow from combining Lemmas 2.2.1 and 2.2.2. To verify (2), we refer to [CGH1, Section 2.3.2] and consider  $\alpha$  of the form

$$\alpha = g(y)d\theta + f(y)dt$$

on a neighborhood  $N(\partial N_{(S,h)})$  of  $\partial N_{(S,h)}$ . Here  $N(\partial N_{(S,h)})$  is the quotient of

$$N(\partial S) \times [0, 1] \simeq [-\varepsilon, 0] \times (\mathbb{R}/\mathbb{Z}) \times [0, 1]$$

with coordinates  $(y, \theta, t)$ , by the equivalence relation  $(y, \theta, 1) \sim (y, \theta, 0)$ . If we take

$$(f(y), g(y)) = (f(0) + y^2/2, g(0) + y),$$

with  $(f(0), g(0))$  in the interior of the first quadrant, then the Reeb vector field  $R_\alpha$  is parallel to

$$-f'(y)\partial_\theta + g'(y)\partial_t = -y\partial_\theta + \partial_t.$$

Its first return map then satisfies (2).  $\square$

From now on, we assume that  $h = h_0$  as given by Lemma 2.1.1.

**2.2.  $ECH(N, \partial N, \alpha)$  and  $\widehat{ECH}(N, \partial N, \alpha)$ .** Let  $N = N_{(S,h)}$  and  $\alpha$  be as in the previous subsection. We recall the definitions of the variants  $ECH(N, \partial N, \alpha)$  and  $\widehat{ECH}(N, \partial N, \alpha)$  and the main result concerning them from [CGH1]. In particular, we carry over the Morse-Bott terminology from [CGH1, Section 5]. We will assume that the almost complex structure  $J$  on  $\mathbb{R} \times N$  is Morse-Bott regular.

The boundary  $\partial N$  is foliated by a Morse-Bott family  $\mathcal{N}$  of simple orbits of  $R_\alpha$  of the form  $\theta = \text{const}$ . We may assume without loss of generality that  $\alpha$  is nondegenerate away from  $\partial N$ , after a  $C^k$ -small perturbation for  $k \gg 0$ . We pick two orbits from  $\mathcal{N}$  and label them  $h$  and  $e$ . The orbits  $h$  and  $e$  are meant to become hyperbolic and elliptic after a small, controlled perturbation of  $\alpha$ . The Morse-Bott family  $\mathcal{N}$  is *negative*. Since  $\mathcal{N}$  is a Morse-Bott family on  $\partial N$ , this means that  $\mathcal{N}$  plays the role of a sink and that no holomorphic curve (besides a trivial cylinder) is asymptotic to an orbit of  $\mathcal{N}$  at the positive end.

**2.2.1.  $ECH(N, \partial N, \alpha)$ .** Let  $\mathcal{P}$  be the set of simple Reeb orbits of  $\alpha$  in  $\text{int}(N)$ . We write  $ECC_j^b(N, \alpha)$  for the chain complex generated by orbit sets  $\gamma$  constructed from  $\mathcal{P} \cup \{e\}$ , whose homology class  $[\gamma]$  intersects the page  $S \times \{t\}$  exactly  $j$  times. The differential for  $ECC_j^b(N, \alpha)$  counts ECH index 1 Morse-Bott buildings in  $(\mathbb{R} \times N, J)$  between orbit sets which are constructed from  $\mathcal{P} \cup \{e\}$ ; for more details see [CGH1, Section 5]. In particular, if  $\tilde{u}$  is an ECH index 1 Morse-Bott building which is counted in the differential, then  $e$  can appear only at a negative

end of  $\tilde{u}$  and no single end of  $\tilde{u}$  can multiply cover  $e$  with multiplicity  $> 1$ . (It is still possible that there are many ends of  $\tilde{u}$  which simply cover  $e$ .)

There are inclusions of chain complexes:

$$\begin{aligned} \mathfrak{I}_j^b : ECC_j^b(N, \alpha) &\rightarrow ECC_{j+1}^b(N, \alpha), \\ \gamma &\mapsto e\gamma, \end{aligned}$$

where we are using multiplicative notation for orbit sets. Let us write  $ECH_j^b(N, \alpha)$  for the homology of the chain complex  $ECC_j^b(N, \alpha)$ . We then define

$$ECH(N, \partial N, \alpha) = \lim_{j \rightarrow \infty} ECH_j^b(N, \alpha).$$

2.2.2.  $\widehat{ECH}(N, \partial N, \alpha)$ . Let  $ECC_j(N, \alpha)$  be the chain complex generated by orbit sets  $\gamma$  constructed from  $\widehat{\mathcal{P}} = \mathcal{P} \cup \{e, h\}$ , whose homology class  $[\gamma]$  intersects  $S \times \{t\}$  exactly  $j$  times. The differential for  $ECC_j(N, \alpha)$  counts ECH index 1 Morse-Bott buildings in  $(\mathbb{R} \times N, J)$ . There are inclusions:

$$\begin{aligned} \mathfrak{I}_j : ECC_j(N, \alpha) &\rightarrow ECC_{j+1}(N, \alpha), \\ \gamma &\mapsto e\gamma, \end{aligned}$$

as before. Writing  $ECH_j(N, \alpha)$  for the homology of  $ECC_j(N, \alpha)$ , we define

$$\widehat{ECH}(N, \partial N, \alpha) = \lim_{j \rightarrow \infty} ECH_j(N, \alpha).$$

The following was the main result of [CGH1]:

**Theorem 2.2.1.** *We have the isomorphisms:*

$$\begin{aligned} ECH(M) &\simeq ECH(N, \partial N, \alpha), \\ \widehat{ECH}(M) &\simeq \widehat{ECH}(N, \partial N, \alpha). \end{aligned}$$

2.3. **Splitting of ECH according to homology classes.** Given an orbit set  $\gamma = \prod_{j=1}^l \gamma_j^{m_j}$  in  $M$  or  $N$ , its *total homology class*  $[\gamma]$  is defined as

$$[\gamma] = \sum_{j=1}^l m_j [\gamma_j],$$

where  $[\gamma_j] \in H_1(M; \mathbb{Z})$  if  $M$  is a closed manifold and  $[\gamma_j] \in H_1(N; \mathbb{Z})$  or  $H_1(N, \partial N; \mathbb{Z})$  (as appropriate) if  $N$  has torus boundary. We then have the direct sum decomposition:

$$ECC(M) = \bigoplus_{A \in H_1(M)} ECC(M, A),$$

where  $ECC(M, A)$  is the subcomplex of  $ECC(M)$  generated by orbit sets with total homology class  $A$ . The direct sum of chain groups descends to a direct sum of homology groups  $ECH(M, A)$ .

There is an analogous splitting for  $ECC(N, \partial N, \alpha)$ . In fact,

$$ECC_j^b(N, \alpha) = \bigoplus_{A \in H_1(N, \partial N)} ECC_j^b(N, \alpha, A),$$

and the inclusion  $\gamma \mapsto e\gamma$  respects this splitting since  $[e] = 0$  in  $H_1(N, \partial N)$ .

**Lemma 2.3.1.** *If  $M$  has an open book decomposition with binding  $K$  and  $N$  is the suspension of a page, then there is an isomorphism*

$$\varpi : H_1(M) \xrightarrow{\sim} H_1(N, \partial N).$$

*Proof.* We use the long exact sequence for the pair  $(M, K)$ :

$$H_1(K) \rightarrow H_1(M) \xrightarrow{i} H_1(M, K) \rightarrow H_0(K).$$

Since  $[K] = 0$  in  $H_1(M)$  and  $K$  is connected, the map  $i$  is an isomorphism. By excision and homotopy invariance, we obtain the isomorphism  $H_1(M, K) \xrightarrow{\sim} H_1(N, \partial N)$ . Combining the two isomorphisms gives us  $\varpi$ .  $\square$

The isomorphism  $ECH(M) \cong ECH(N, \partial N, \alpha)$  respects the splitting into total homology classes. In fact

$$ECH(M, A) \cong ECH(N, \partial N, \alpha, \varpi(A)).$$

The same also holds for the hat versions.

**2.4. Twisted coefficients in ECH.** In this subsection we describe the construction of ECH with twisted coefficients. We will adapt the analogous construction in Heegaard Floer homology from [OSz2, Section 8] instead of following the original construction in [HS2, Section 11].

Fix a homology class  $A$  and a closed curve  $\Gamma \subset M$  such that  $[\gamma] = A$ . Let  $\gamma^+$  and  $\gamma^-$  be orbit sets with  $[\gamma^+] = [\gamma^-] = A$ . We denote by  $H_2(M, \gamma^+, \gamma^-)$  the set of relative homology classes of 2-chains  $C$  in  $M$  with  $\partial C = \gamma^+ - \gamma^-$  and by  $\mathcal{M}^{I=1}(\gamma^+, \gamma^-, C)$  the moduli spaces of  $I = 1$  holomorphic curves in  $\mathbb{R} \times M$  from  $\gamma^+$  to  $\gamma^-$  representing the homology class  $C$ .

A complete set of paths for  $A$  based at  $\Gamma$  is the choice, for every orbit set  $\gamma$  such that  $[\gamma] = A$ , of a surface  $C_\gamma \subset M$  such that  $\partial C_\gamma = \gamma - \Gamma$ . A complete set of paths for  $A$  induces maps

$$\mathfrak{A}' : H_2(M, \gamma^+, \gamma^-) \rightarrow H_2(M)$$

for all  $\gamma^+$  and  $\gamma^-$  in  $A$  by  $\mathfrak{A}'(C) = [C_{\gamma^-} \cup C \cup -C_{\gamma^+}]$ . This map is compatible with the action of  $H_2(M)$  on  $H_2(M, \gamma^+, \gamma^-)$  and with the concatenation of chains with matching ends.

We denote the group ring of  $H_2(M; \mathbb{Z})$  by  $\mathbb{F}[H_2(M; \mathbb{Z})]$  and the generator corresponding to  $c \in H_2(M; \mathbb{Z})$  by  $e^c$ . We define

$$\underline{ECC}(M, \alpha, A) = ECC(M, \alpha, A) \otimes_{\mathbb{F}} \mathbb{F}[H_2(M; \mathbb{Z})]$$

as an  $\mathbb{F}[H_2(M; \mathbb{Z})]$ -module, with differential

$$\partial\gamma^+ = \sum_{\gamma^-} \sum_{C \in H_2(M, \gamma^+, \gamma^-)} \#(\mathcal{M}^{I=1}(\gamma^+, \gamma^-, C)/\mathbb{R}) e^{\mathfrak{A}'(C)} \gamma^-.$$

The homology of this complex is ECH with twisted coefficients  $\underline{ECH}(M, A)$ . The  $U$ -map can be defined in a similar manner and  $\widehat{\underline{ECH}}(M, A)$  is the homology



of its mapping cone. The construction of  $\underline{ECH}(N, \partial N, \varpi(A))$  with coefficient ring  $\mathbb{F}[H_2(N, e; \mathbb{Z})] \cong \mathbb{F}[H_2(M; \mathbb{Z})]$  is similar, and there are isomorphisms

$$(2.4.1) \quad \underline{ECH}(M, A) \simeq \underline{ECH}(N, \partial N, \varpi(A)).$$

$$(2.4.2) \quad \widehat{\underline{ECH}}(M, A) \simeq \widehat{\underline{ECH}}(N, \partial N, \varpi(A)).$$

Moreover, by considering only orbit sets that intersect a page  $j$  times we can define  $\underline{ECH}_j(N)$  and we have

$$\widehat{\underline{ECH}}(N, \partial N, \alpha, \varpi(A)) = \lim_{j \rightarrow \infty} \underline{ECH}_j(N, \varpi A).$$

**2.5. Elimination of elliptic orbits.** The goal of this subsection is to show how to locally replace elliptic orbits by hyperbolic orbits with the same parity (i.e., with negative eigenvalues). The main result, which is used in [CGH2] but is also of independent interest, is Theorem 2.5.2 below. Let us first give the following definition:

**Definition 2.5.1** (Filtration  $\mathcal{F}$ ). If  $N = N_{(S,h)}$  is the suspension of  $(S, h)$  and  $\gamma \subset N$  is a link which is everywhere transverse to  $S \times \{t\}$ ,  $t \in [0, 1]$ , then we define  $\mathcal{F}(\gamma) = \langle \gamma, S \times \{0\} \rangle$ .

**Theorem 2.5.2** (Elimination of elliptic orbits). *Let  $\alpha$  be a contact form on the suspension  $N = N_{(S,h)}$ ,  $h \in \text{Diff}(S, \partial S)$ , such that the Reeb vector field  $R_\alpha$  is transverse to  $S \times \{t\}$ ,  $t \in [0, 1]$ , and  $h$  is the first return map of  $R_\alpha$  on  $S \times \{0\}$ . Then, given  $m \in \mathbb{Z}^+$  and  $\varepsilon > 0$ , there exists a smooth function  $f : N \rightarrow (0, +\infty)$  which is  $\varepsilon$ -close to 1 with respect to a fixed  $C^1$ -norm and whose Reeb vector field  $R_{f\alpha}$  has no elliptic orbits  $\gamma$  in  $\text{int}(N)$  satisfying  $\mathcal{F}(\gamma) \leq m$ .*

**2.5.1. Model situation on the solid torus.** Fix a constant  $\delta > 0$ . Consider the solid torus

$$V = D^2 \times S^1 = D^2 \times (\mathbb{R}/\mathbb{Z}) = \{(r, \theta, z) \mid r \leq \delta\}$$

with the contact structure  $\xi_0 = \ker \alpha_0$ , where  $\alpha_0 = dz + r^2 d\theta$ . We write  $D_{z_0} = \{z = z_0\} \subset V$  and  $T_{r_0} = \{r = r_0\} \subset V$ .

Given a function  $f : [0, \delta] \rightarrow (0, +\infty)$ , the Reeb vector field  $R_{f(r)\alpha_0}$  is given by:

$$(2.5.1) \quad R_{f\alpha_0} = \frac{1}{2rf}((r^2 f' + 2rf)\partial_z - f'\partial_\theta).$$

In particular,  $R_{f\alpha_0}$  is transverse to each  $D_z$  and  $R_{f\alpha_0}$  is tangent to each  $T_r$ . The first return map  $\Phi_{f\alpha_0} : D_0 \xrightarrow{\sim} D_0$  is a rotation on each circle  $\{r = r_0\}$  and can be written as  $(r, \theta) \mapsto (r, \theta + \phi_{f\alpha_0}(r))$ .

**Claim 2.5.3.** *Let  $C : [0, \delta] \rightarrow (0, +\infty)$  be a smooth function. If*

$$(2.5.2) \quad f(r) = A \exp \left( - \int_0^r \frac{2sC(s)}{1 + C(s)s^2} ds \right),$$

*then the first return map of  $R_{f\alpha_0}$  satisfies  $\phi_{f\alpha_0}(r) = C(r)$  for each  $r \in (0, \delta]$ .*

*Proof.* In view of Equation (2.5.1),  $R_{f\alpha_0}$  is parallel to  $\partial_z + C(r)\partial_\theta$  if and only if

$$(2.5.3) \quad C(r)(r^2 f' + 2rf) = -f'$$

is satisfied.  $\square$

Let  $f_0 : [0, \delta] \rightarrow (0, +\infty)$  be a function such that  $\phi_{f_0\alpha_0}(r) = \phi_0$ , where  $\phi_0 \in (0, 2\pi)$ . By Claim 2.5.3, we may take

$$f_0(r) = \exp\left(-\int_\delta^r \frac{2s\phi_0}{1+\phi_0 s^2} ds\right) = \frac{1+\phi_0\delta^2}{1+\phi_0 r^2}.$$

In particular,  $\gamma_0 = \{0\} \times S^1$  is the only orbit  $\gamma$  of  $R_{f_0\alpha_0}$  satisfying  $\mathcal{F}(\gamma) = 1$ , where  $\mathcal{F}(\gamma) = \langle \gamma, D_0 \rangle$ , and the orbit  $\gamma_0$  is elliptic and nondegenerate.

**Lemma 2.5.4** (Modification lemma). *There exists a function  $f_2 : V \rightarrow (0, +\infty)$  such that  $f_2\alpha_0$  is arbitrarily  $C^1$ -close to  $f_0\alpha_0$  and the Reeb vector field  $R_{f_2\alpha_0}$  is equal to  $R_{f_0\alpha_0}$  near  $\partial V$ , is transverse to  $D_z$  for all  $z \in S^1$ , and has only one orbit  $\gamma$  satisfying  $\mathcal{F}(\gamma) = 1$ . Moreover the orbit  $\gamma$  is hyperbolic.*

*Proof.* Without loss of generality we take  $\phi_0 \in (0, \pi]$ ; the case  $\phi_0 \in [\pi, 2\pi)$  is similar.

Let  $0 < \delta' \ll \delta$  and let  $C_{\delta'} : [0, \delta] \rightarrow [\phi_0, \pi]$  be a smooth function such that:

- $C_{\delta'}(r) = \pi$  on  $[0, \frac{\delta'}{2}]$ ; and
- $C_{\delta'}(r) = \phi_0$  on  $[\delta', \delta]$ .

Let  $f_1 : [0, \delta] \rightarrow (0, +\infty)$  be the smooth function

$$f_1(r) = \exp\left(-\int_\delta^r \frac{2sC_{\delta'}(s)}{1+C_{\delta'}(s)s^2} ds\right).$$

The function  $f_1$  satisfies the following:

- (1)  $f_1 = f_0$  and  $R_{f_1\alpha_0} = R_{f_0\alpha_0}$  for  $r \in [\delta', \delta]$ ;
- (2)  $R_{f_1\alpha_0} \lrcorner D_z$  for all  $z \in S^1$ ;
- (3)  $\phi_{f_1\alpha_0}(r) = C_{\delta'}(r)$  for all  $r \in [0, \delta]$ .

We compute that

$$(2.5.4) \quad \frac{f_1(r)}{f_0(r)} = \exp\left(\int_\delta^r \left(\frac{2s\phi_0}{1+\phi_0 s^2} - \frac{2sC_{\delta'}(s)}{1+C_{\delta'}(s)s^2}\right) ds\right).$$

By taking  $\delta' > 0$  to be arbitrarily small, the absolute value of the integrand of Equation (2.5.4) can be made arbitrarily small. Hence  $f_1(r) \approx f_0(r)$  for all  $r \in [0, \delta]$ , in view of (1).

Next we consider  $\partial_x f_i = \frac{x f'_i(r)}{r}$  and  $\partial_y f_i = \frac{y f'_i(r)}{r}$ , where  $r = \sqrt{x^2 + y^2}$ . We appeal to Equation (2.5.3) and write

$$f'_1(r) = \frac{2r f_1(r)}{1 + C_{\delta'}(r)r^2}, \quad f'_0(r) = \frac{2r f_0(r)}{1 + \phi_0 r^2}.$$

For  $\delta' > 0$  sufficiently small,  $f'_1(r)$  and  $f'_0(r)$  are both arbitrarily close to 0. Moreover  $f'_1(r) = f'_0(r)$  for  $r \in [\delta', \delta]$ . Hence  $f'_1 \approx f'_0$  for all  $r \in [0, \delta]$ . This implies that  $\partial_* f_1 \approx \partial_* f_0$  and  $\partial_*(f_1/f_0) \approx 0$  for  $* = x, y$ .

The Reeb vector field  $R_{f_1\alpha_0}$  has exactly one orbit  $\gamma_0$  satisfying  $\mathcal{F}(\gamma_0) = 1$ . The linearized first return map  $d\Phi_{f_1\alpha_0}(0)$  of  $\gamma_0$  has eigenvalues  $-1$ .

We now claim there exists a  $C^k$ -small perturbation  $f_2$  of  $f_1$  for any  $k \gg 0$  such that the linearized first return map  $d\Phi_{f_2\alpha_0}(0)$  of the corresponding orbit has eigenvalues  $-\lambda$  and  $-\frac{1}{\lambda}$  with  $\lambda \in \mathbb{R}^{>0} - \{1\}$ . This follows from a local model on  $D^2 \times [0, 1]$  with coordinates  $(x, y, z)$ : Suppose the Reeb vector field is  $R = \partial_z$ . Then the contact 1-form can be written as  $\alpha = dz + \beta$ , where  $\beta$  is a 1-form on  $D^2$ . We consider  $R_{h\alpha}$ , where  $h(x, y) = x^2 - y^2$ . The component of  $R_{h\alpha}$  in the  $xy$ -direction is parallel to  $y\partial_x - x\partial_y$ . Hence the derivative at zero of the holonomy map  $D^2 \times \{0\} \dashrightarrow D^2 \times \{1\}$ ,<sup>1</sup> obtained by flowing along  $R_{h\alpha}$ , has eigenvalues  $\lambda_0$  and  $\frac{1}{\lambda_0}$  with  $\lambda_0 \in \mathbb{R}^{>0} - \{1\}$ . By appropriately damping  $\varepsilon h$  out to zero outside a small neighborhood of  $(0, 0, 0) \in D^2 \times [0, 1]$  (here  $\varepsilon > 0$  is sufficiently small), the above model can be grafted into  $V$  to give  $f_2$ . This procedure does not introduce any extra  $\mathcal{F} = 1$  orbits, since the graph of  $\Phi_{f_1\alpha_0}$  in  $D_0 \times D_0$  intersects the diagonal transversely in one point and this property is stable under a  $C^k$ -small perturbation of  $\Phi_{f_1\alpha_0}$  for any  $k \gg 0$ .  $\square$

*Remark 2.5.5.* The modification in Lemma 2.5.4 introduces many elliptic orbits satisfying  $\mathcal{F} > 1$ .

*2.5.2. Proof of Theorem 2.5.2.* We now prove Theorem 2.5.2. Starting with the contact form  $\alpha$  on  $N$ , we make a  $C^2$ -small perturbation of  $\alpha$  relative to  $\partial N$  such that the resulting Reeb vector field — also called  $R_\alpha$  — satisfies the following:

- (1)  $R_\alpha$  is nondegenerate away from  $\partial N$ ;
- (2) for each  $\mathcal{F} = 1$  elliptic orbit  $\gamma \subset \text{int}(N)$ , the first return map is a rotation by an irrational angle and  $\alpha$  is of the form  $C_0 f_0 \alpha_0$  on a tubular neighborhood  $V_\gamma$  of  $\gamma$ .

Here  $f_0$  and  $\alpha_0$  are as in Section 2.5.1 and  $C_0$  is some constant. We then use Lemma 2.5.4 on the tubular neighborhoods  $V_\gamma$  to replace the  $\mathcal{F} = 1$  elliptic orbits by hyperbolic orbits plus  $\mathcal{F} > 1$  orbits. Next, we perturb the form so that the  $\mathcal{F} = 1$  orbits and the  $\mathcal{F} = 2$  hyperbolic orbits are left unchanged and the  $\mathcal{F} = 2$  elliptic orbits satisfy (2), with  $\mathcal{F} = 1$  replaced by  $\mathcal{F} = 2$ . Using Lemma 2.5.4 again, we replace the  $\mathcal{F} = 2$  elliptic orbits by hyperbolic orbits and  $\mathcal{F} > 2$  orbits. Continuing in this manner, we obtain  $\alpha$  without any  $\mathcal{F} \leq m$  elliptic orbits in  $\text{int}(N)$ .

*2.5.3. Direct limits.* Starting with  $(S, h)$  and  $\alpha$  from Section 2.1, we define  $f_j : N \rightarrow (0, +\infty)$ ,  $j \in \mathbb{N}$ , inductively as follows. Let  $f_0 = 1$ . Suppose we have chosen up to  $f_j$  so that  $R_{f_j\alpha}$  has no elliptic orbits with  $\mathcal{F} \leq j$ . Let  $V_{j+1}$  be a small tubular neighborhood of the elliptic orbits of  $R_{f_j\alpha}$  with  $\mathcal{F} = j + 1$ . Then we choose  $f_{j+1}$  such that the following hold:

- (1)  $f_{j+1}$  is  $C^1$ -close to  $f_j$ ;
- (2)  $f_{j+1} = f_j$  on  $N - V_{j+1}$ ;
- (3)  $R_{f_{j+1}\alpha}$  has no elliptic orbits with  $\mathcal{F} \leq j$ .

---

<sup>1</sup>The dashed arrow indicates that the map is only partially defined.

The existence of  $f_{j+1}$  is given by Theorem 2.5.2.

Next we consider the ECH chain maps

$$(2.5.5) \quad \mathfrak{J}_j : ECC_j(N, f_j \alpha) \rightarrow ECC_{j+1}(N, f_{j+1} \alpha),$$

given by composing two maps:

$$\mathfrak{J}'_j : ECC_j(N, f_j \alpha) \rightarrow ECC_j(N, f_{j+1} \alpha)$$

and  $ECC_j(N, f_{j+1} \alpha) \rightarrow ECC_{j+1}(N, f_{j+1} \alpha)$  given by  $\gamma \mapsto e\gamma$ . The map  $\mathfrak{J}'_j$  is defined by suitably completing  $f_j \alpha$  and  $f_{j+1} \alpha$  to  $M$  and applying the ECH cobordism map given by [HT3, Theorem 2.4]. It is important to remember that the ECH cobordism map is defined through Seiberg-Witten Floer cohomology.

Then we have:

**Theorem 2.5.6.**  $\widehat{ECH}(M) \simeq \lim_{j \rightarrow \infty} ECH_j(N, f_j \alpha)$ , where direct limit is taken with respect to the maps  $\mathfrak{J}_j$ .

The proof is omitted, since it is similar to that of Theorem 1.2.1.

### 3. PERIODIC FLOER HOMOLOGY

In order to simplify some technicalities, we would like to replace the ECH groups by the *periodic Floer homology groups* of Hutchings [Hu1, Hu2], abbreviated PFH in this paper. The PFH groups are defined in a manner completely analogous to the ECH groups, with stable Hamiltonian vector fields replacing the Reeb vector fields.

If  $M$  is a closed manifold which fibers over the circle, then the PFH groups of  $M$  are equivalent to the Seiberg-Witten Floer cohomology groups of  $M$  by the work of Lee-Taubes [LT].

**3.1. Interpolating between Reeb and stable Hamiltonian vector fields.** Consider the contact form  $\alpha = f_t dt + \beta_t$  on  $S \times [0, 1]$ , as defined in Section 2.1. We may assume that  $R_\alpha$  is parallel to  $\partial_t$  on  $S \times [0, 1]$ . Since

$$d\alpha = d_S f_t \wedge dt + d_S \beta_t + dt \wedge \frac{d\beta_t}{dt},$$

where  $d_S$  is the exterior derivative in the  $S$ -direction, it follows that  $\frac{d\beta_t}{dt} = -d_S f_t$ . (Hence  $d_S \beta_t$  is an area form which does not depend on  $t$ .) Also, the form  $\alpha$  is a contact form as long as  $d_S \beta_t > 0$ . Hence, for  $C \gg 0$ , the form  $(C + f_t)dt + \beta_t$  is a contact form with Reeb vector field parallel to  $\partial_t$ .

Now consider the 1-parameter family of 1-forms

$$(3.1.1) \quad \alpha_\tau = Cdt + \tau(f_t dt + \beta_t),$$

$\tau \in [0, 1]$ , on  $N$ . It interpolates between the contact form  $\alpha_1 = Cdt + (f_t dt + \beta_t)$  and the stable Hamiltonian form  $\alpha_0 = Cdt$ . The Reeb vector fields  $R_\tau = R_{\alpha_\tau}$  are directed by  $\partial_t$  and hence are parallel for all  $\tau > 0$ .

The pair  $(\alpha_\tau, \omega = d_S \beta_t)$  is a *stable Hamiltonian structure* on  $N$ . When  $\tau = 0$ , the Hamiltonian vector field  $R_0$  equals  $\frac{1}{C}\partial_t$  and hence is parallel to all the  $R_\tau$ ,  $\tau > 0$ . Also let  $\xi_\tau$  be the 2-plane field on  $N$  given by the kernel of  $\alpha_\tau$ . The

closed 2-form  $\omega$  can either be viewed as an area form on  $S$  or as a (maximally nondegenerate) 2-form on  $N$ .

**3.2. Definitions.** Consider the infinite cylinder  $\mathbb{R} \times N$  with coordinates  $(s, x)$ .

**Definition 3.2.1.** An almost complex structure  $J_\tau$  on  $\mathbb{R} \times N$  is *adapted to the stable Hamiltonian structure*  $(\alpha_\tau, \omega)$  if  $J_\tau$  satisfies the following:

- $J_\tau$  is  $s$ -invariant;
- $J_\tau(\partial_s) = R_\tau$  and  $J_\tau(\xi_\tau) = \xi_\tau$ ;
- $J_\tau$  is tamed by the symplectic form  $\Omega = ds \wedge dt + \omega$ .

Our goal is to replace the ECH chain complexes

$$ECC_j(N, \alpha_\tau, J_\tau), \quad ECC_j^b(N, \alpha_\tau, J_\tau)$$

for  $\tau > 0$ , by the analogously defined PFH chain complexes

$$PFC_j(N, \alpha_0, \omega, J_0), \quad PFH_j^b(N, \alpha_0, \omega, J_0).$$

The orbit sets of the ECH chain groups are constructed using the Reeb vector fields  $R_\tau$  and the orbit sets of the PFH chain groups are constructed using the Hamiltonian vector field  $R_0$ .

We introduce some notation. Let  $J_\tau, \tau \in [0, 1]$ , be a smooth family of  $(\alpha_\tau, \omega)$ -adapted almost complex structures, let  $\gamma, \gamma'$  be orbit sets of  $PFC_j(N, \alpha_0, \omega)$  or  $ECC_j(N, \alpha_\tau)$ , and let  $Z \in H_2(N, \gamma, \gamma')$ . We write:

- $\mathcal{M}_{J_\tau}(\gamma, \gamma')$  for the moduli space of  $J_\tau$ -holomorphic curves from  $\gamma$  to  $\gamma'$ ;
- $\mathcal{M}_{J_\tau}(\gamma, \gamma', Z) \subset \mathcal{M}_{J_\tau}(\gamma, \gamma')$  for the subset of curves in the class  $Z$ ;
- $\mathcal{M}_{J_\tau}^s(\gamma, \gamma') \subset \mathcal{M}_{J_\tau}(\gamma, \gamma')$  for the subset of simply-covered curves; and
- $\mathcal{M}'_{J_\tau}(\gamma, \gamma', Z) \subset \mathcal{M}_{J_\tau}(\gamma, \gamma', Z)$  for the subset of curves without connector components.

Here a *connector over an orbit set*  $\delta = \prod_i \delta_i^{m_i}$  is a collection of branched covers of trivial cylinders where the branching is optional and the total covering degree over  $\mathbb{R} \times \delta_i$  is  $m_i$ .

Given two orbit sets  $\delta = \prod_i \delta_i^{m_i}$  and  $\delta' = \prod_i \delta_i^{m'_i}$ , we set  $\delta/\delta' = \prod_i \delta_i^{m_i - m'_i}$  if  $m'_i \leq m_i$  for all  $i$ ; otherwise we set  $\delta/\delta' = 0$ . Here some  $m_i$  and  $m'_i$  may be zero. We then write  $u \in \mathcal{M}_{J_\tau}(\gamma, \gamma', Z)$  as  $u^0 \cup u^1$ , where  $u^0$  is a connector over the orbit set  $\gamma_0$  and  $u^1 \in \mathcal{M}'_{J_\tau}(\gamma/\gamma_0, \gamma'/\gamma_0)$ . By [HT1, Proposition 7.5], if  $I_{ECH}(\gamma, \gamma', Z) = 1$  (resp. 2), then  $u^1 \in \mathcal{M}_{J_\tau}^s(\gamma/\gamma_0, \gamma'/\gamma_0)$  and has Fredholm index 1 (resp. 2).

**3.3. The flux.** For more details, see for example [CHL, Section 2]. Let

$$h_* : H_1(S; \mathbb{Z}) \rightarrow H_1(S; \mathbb{Z})$$

be the map on homology induced by  $h : (S, \omega) \xrightarrow{\sim} (S, \omega)$  and let  $K$  be the kernel of  $h_* - id$ . Then the *flux*  $F_h : K \rightarrow \mathbb{R}$  is defined as follows: Let  $[\delta] \in K$ . Since  $[\delta] = [h(\delta)] \in H_1(S; \mathbb{Z})$ , there is a 2-chain  $\mathcal{S} \subset S$  such that  $\partial \mathcal{S} = \delta - h(\delta)$ . We then define:

$$F_h(\delta) = \int_{\mathcal{S}} \omega.$$

Since the map  $h$  is the first return map of a Reeb vector field  $R_\tau$ , it has zero flux (cf. [CHL, Lemma 2.2]). This implies that  $\int_{[C]} \omega = 0$  for every  $[C] \in H_2(N)$ , where  $\omega$  is now viewed as a closed 2-form on  $N$ . Indeed,  $[C]$  can be represented by a surface of the form  $(\delta \times [0, 1]) \cup (\mathcal{S} \times \{0\})$ , where the relevant boundary components are glued. Hence, if  $\gamma$  and  $\gamma'$  are orbit sets of  $PF C_j(N, \alpha_0, \omega_0, J_0)$ , then the  $\omega$ -area of any  $Z \in H_2(N, \gamma, \gamma')$  only depends on  $\gamma$  and  $\gamma'$ .

**3.4. Compactness.** The vanishing of the flux is an important ingredient in establishing that  $\mathcal{M}_{J_0}(\gamma, \gamma')$  admits a compactification in the sense of [BEHWZ].

We briefly outline the argument from [Hu1, Section 9]: (1) There is a bound on the  $\omega$ -area for all elements of  $\mathcal{M}_{J_0}(\gamma, \gamma')$ . This was done above. (2) Given a sequence of holomorphic curves  $u_i \in \mathcal{M}_{J_0}(\gamma, \gamma')$ ,  $i \in \mathbb{N}$ , there is a subsequence which converges weakly as currents to a holomorphic building  $u_\infty$ . This is due to the Gromov-Taubes compactness theorem [T3], which works in dimension four and does not require any a priori bound on the genus of  $u_i$ . The extraction of the holomorphic building is treated in some detail in [Hu1, Lemma 9.8]. Hence we may assume that the homology classes  $[u_i] \in H_2(N, \gamma, \gamma')$  are fixed. (3) There is a bound on the genus of the curve, provided the homology classes  $[u_i]$  are fixed. This follows from the adjunction inequality and will be discussed below. (4) Once there is a genus bound, apply the SFT compactness theorem of [BEHWZ].

We now explain in some detail how to obtain genus bounds from bounds on the homology classes  $[u_i]$ , especially since similar arguments will appear in later sections. But first let us introduce some notation.

Let  $(F, j)$  be a closed Riemann surface and  $\mathbf{p}^+$  and  $\mathbf{p}^-$  be disjoint finite sets of punctures of  $F$ . Then let

$$u : \dot{F} = F - \mathbf{p}^+ - \mathbf{p}^- \rightarrow \mathbb{R} \times N,$$

be a  $(j, J_0)$ -holomorphic map from  $\gamma = \prod_i \gamma_i^{m_i}$  to  $\gamma' = \prod_i \gamma_i^{n_i}$ . Here the punctures of  $\mathbf{p}^\pm$  are asymptotic to the  $\pm$  ends of  $u$ . The positive ends of  $u$  partition  $m_i$  into  $(m_{i1}, m_{i2}, \dots)$  and the negative ends of  $u$  partition  $n_i$  into  $(n_{i1}, n_{i2}, \dots)$ . (We ignore the partition terms that are zero.) Pick a trivialization  $\tau$  of  $TS$  in a neighborhood of all the  $\gamma_i$ , and let  $\mu_\tau(\gamma_i^k)$  be the usual Conley-Zehnder index of the  $k$ -fold cover of  $\gamma_i$  with respect to  $\tau$ .<sup>2</sup> Then we define the *total Conley-Zehnder indices* at the positive and negative ends of  $u$  as follows:

$$\mu_\tau^+(u) = \sum_i \sum_r \mu_\tau(\gamma_i^{m_{ir}}),$$

$$\mu_\tau^-(u) = \sum_i \sum_r \mu_\tau(\gamma_i^{n_{ir}}),$$

---

<sup>2</sup>The trivialization  $\tau$ , used in Section 3.4, is not to be confused with the parameter  $\tau$ , used in the rest of Section 3.

and also write  $\mu_\tau(u) = \mu_\tau^+(u) - \mu_\tau^-(u)$ . The *symmetric Conley-Zehnder index* of Hutchings [Hu1], so called because of its motivation from studying the “symplectomorphism”  $Sym^k(h)$  of  $Sym^k(S)$  induced by  $h$ ,<sup>3</sup> is defined as:

$$\tilde{\mu}_\tau(\gamma) = \sum_i \sum_{r=1}^{m_i} \mu_\tau(\gamma_i^r),$$

and does not depend on the choice of  $u$  from  $\gamma$  to  $\gamma'$ . We write  $\tilde{\mu}_\tau(u) = \tilde{\mu}_\tau(\gamma) - \tilde{\mu}_\tau(\gamma')$ . We also recall the *writhe*

$$w_\tau(u) = w_\tau^+(u) - w_\tau^-(u),$$

where  $w_\tau^+(u)$  is the total writhe of braids  $u(\dot{F}) \cap (\{s\} \times N)$ ,  $s \gg 0$ , viewed in the union of solid torus neighborhoods of  $\gamma_i$  and computed with respect to the framing  $\tau$ ; and  $w_\tau^-(u)$  is defined similarly.

The key ingredient in establishing genus bounds is the relative adjunction formula from [Hu1, Equation (18)] for simple curves  $u$  with a finite number of singularities and no connector components:

$$(3.4.1) \quad c_1(u^*TS, \tau) = \chi(\dot{F}) + w_\tau(u) + Q_\tau(u) - 2\delta(u),$$

where  $Q_\tau(u)$  is the relative intersection pairing with respect to  $\tau$  and  $\delta(u)$  is a nonnegative integer which is a count of the singularities. In particular,  $\delta(u) = 0$  if and only if  $u$  is an embedding (see [M1, MW]). Together with the writhe bounds

$$(3.4.2) \quad \begin{aligned} w_\tau^+(u) &\leq \tilde{\mu}_\tau(\gamma) - \mu_\tau^+(u), \\ w_\tau^-(u) &\geq \tilde{\mu}_\tau(\gamma') - \mu_\tau^-(u), \end{aligned}$$

from [Hu2, Lemma 4.20], we obtain:

$$(3.4.3) \quad \chi(\dot{F}) \geq c_1(u^*TS, \tau) + \mu_\tau(u) - \tilde{\mu}_\tau(u) - Q_\tau(u) + 2\delta(u).$$

(See [Hu1, Theorem 10.1].) Since all of the terms on the right-hand side are either homological quantities or depend on the data near  $\gamma$  and  $\gamma'$ , we have a lower bound on  $\chi(\dot{F})$ , which implies an upper bound on the genus of  $\dot{F}$ .

**3.5. Transversality.** Let  $\mathcal{J}_\tau$  be the space of almost complex structures  $J_\tau$  on  $\mathbb{R} \times N$  in the class  $C^\infty$  which are adapted to  $(\alpha_\tau, \omega)$ .

**Definition 3.5.1.** An almost complex structure  $J_\tau \in \mathcal{J}_\tau$  is *j-regular* if, for all orbit sets  $\gamma$  and  $\gamma'$  of  $R_\tau$  which intersect  $S \times \{0\}$  at most  $j$  times, the moduli space  $\mathcal{M}_{J_\tau}^s(\gamma, \gamma')$  is transversely cut out.

Let  $\mathcal{J}_\tau^{reg, j} \subset \mathcal{J}_\tau$  be the subset of  $j$ -regular  $J_\tau$ . The following lemma states that  $\mathcal{J}_0^{reg, j} \subset \mathcal{J}_0$  is dense.

**Lemma 3.5.2** (Transversality). *A generic  $J_0 \in \mathcal{J}_0$  is  $j$ -regular.*

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<sup>3</sup> $Sym^k(h)$  is a symplectomorphism away from the (multi)-diagonal.

*Proof.* This follows from [Hu1, Lemma 9.12(b)], which states that a generic  $J_0 \in \mathcal{J}_0$  is regular away from holomorphic curves which have a fiber  $\{(s, t)\} \times S$  as an irreducible component. (Observe that the fibers are holomorphic for any  $J_0 \in \mathcal{J}_0$ .) In our case, the fibers are not closed and cannot occur as irreducible components of curves in  $\mathcal{M}_{J_0}^s(\gamma, \gamma')$ .  $\square$

**3.6. The equivalence of certain ECH and PFH groups.** In this section we prove the following theorem:

**Theorem 3.6.1.** *Given  $j > 0$ , there exist  $J_0 \in \mathcal{J}_0^{reg,j}$  and  $\tau_0 = \tau_0(j, J_0) > 0$  such that there are isomorphisms of chain complexes*

$$PFC_j(N, \alpha_0, \omega, J_0) \simeq ECC_j(N, \alpha_\tau, J_\tau),$$

$$PFC_j^\flat(N, \alpha_0, \omega, J_0) \simeq ECC_j^\flat(N, \alpha_\tau, J_\tau),$$

for all  $0 < \tau \leq \tau_0$ . Here  $J_\tau \in \mathcal{J}_\tau^{reg,j}$  is sufficiently close to  $J_0$ . Similar isomorphisms hold with twisted coefficients.

*Proof.* We will prove the first equivalence, leaving the second to the reader.

Since there is a one-to-one correspondence between the generators of the chain groups  $PFC_j(N, \alpha_0, \omega)$  and  $ECC_j(N, \alpha_\tau)$ , we have

$$PFC_j(N, \alpha_0, \omega) \simeq ECC_j(N, \alpha_\tau)$$

as  $\mathbb{F}$ -vector spaces, but not necessarily as chain complexes. In other words, we may view any orbit set  $\gamma$  for  $R_\tau$  as an orbit set of any other  $R_{\tau'}$ .

Let  $J_0$  be an almost complex structure in  $\mathcal{J}_0^{reg,j}$ . By Lemma 3.5.2,  $\mathcal{J}_0^{reg,j}$  is a dense subset of  $\mathcal{J}_0$ , and in particular is nonempty. The moduli spaces  $\mathcal{M}'_{J_0}(\gamma, \gamma', Z)$  of ECH index 1 and 2 are transversely cut out since they are simple by the ECH index inequality. Now let  $J_\tau, \tau \in [0, 1]$ , be a smooth family of  $(\alpha_\tau, \omega)$ -adapted almost complex structures which extend  $J_0$ .

We claim that the ECH index 1 moduli spaces  $\mathcal{M}'_{J_\tau}(\gamma, \gamma', Z)$  are transversely cut out and diffeomorphic to  $\mathcal{M}'_{J_0}(\gamma, \gamma', Z)$ , when  $\tau > 0$  is sufficiently small. Indeed, if  $u_\tau \in \mathcal{M}'_{J_\tau}(\gamma, \gamma', Z)$  is sufficiently close to  $\mathcal{M}'_{J_0}(\gamma, \gamma', Z)$ , then the moduli space is regular at  $u_\tau$ . Hence it suffices to prove that, if  $\tau > 0$  is sufficiently small, then every  $u_\tau \in \mathcal{M}'_{J_\tau}(\gamma, \gamma', Z)$  is sufficiently close to  $\mathcal{M}'_{J_0}(\gamma, \gamma', Z)$ . Indeed, this follows from the compactness argument from Section 3.4. Let  $u_i \in \mathcal{M}'_{J_{\tau_i}}(\gamma, \gamma', Z)$  be a sequence of ECH index 1 holomorphic curves with  $\tau_i \rightarrow 0$ . By the compactness theorem and incoming/outgoing partition considerations,  $u_i$  converges to  $u \in \mathcal{M}_{J_0}(\gamma, \gamma', Z)$  with  $I_{ECH}(u) = 1$ , after possibly taking a subsequence. In particular, the limit  $u$  is not a holomorphic building with multiple levels. If  $u$  has connector components  $u^0$  over  $\gamma_0$ , then  $u_i$  must also have connector components  $u_i^0$  over  $\gamma_0$ , a contradiction. Hence  $u \in \mathcal{M}'_{J_0}(\gamma, \gamma', Z)$ , which proves the claim.

Since the chain groups are isomorphic as vector spaces and the differentials agree for sufficiently small  $\tau > 0$ , the theorem follows.  $\square$



#### 4. A VARIATION OF $\widehat{HF}(M)$ ADAPTED TO OPEN BOOK DECOMPOSITIONS

In this section we recall the cylindrical reformulation of Heegaard Floer homology. This reformulation was suggested by Eliashberg and worked out in detail by Lipshitz [Li]. The discussion is slightly different from that of [Li] in that we introduce an ECH-type index  $I_{HF}$  and define the Heegaard Floer groups in terms of  $I_{HF}$ . We then use the work of [HKM1] to restrict the Heegaard Floer data to the page  $S$  of an open book decomposition.

**4.1. Heegaard data.** A *pointed Heegaard diagram* is a quadruple  $(\Sigma, \alpha, \beta, z)$  which consists of the following:

- a closed oriented surface  $\Sigma$  of genus  $k$ ;
- two collections  $\alpha = \{\alpha_1, \dots, \alpha_k\}$  and  $\beta = \{\beta_1, \dots, \beta_k\}$  of  $k$  pairwise disjoint simple closed curves in  $\Sigma$ ; and
- a point  $z \in \Sigma - \alpha - \beta$ ;

where each of  $\alpha$  and  $\beta$  forms a basis of  $H_1(\Sigma; \mathbb{Z})$  and  $\alpha$  and  $\beta$  intersect transversely in  $\Sigma$ .

If  $M$  is a closed oriented 3-manifold, then  $(\Sigma, \alpha, \beta, z)$  is a *pointed Heegaard diagram for  $M$*  if  $\Sigma$  decomposes  $M$  into two handlebodies, i.e.,  $M = H_\alpha \cup_\Sigma H_\beta$  and  $\Sigma = \partial H_\alpha = -\partial H_\beta$ , where the  $\alpha$ -curves (resp.  $\beta$ -curves) bound compression disks in the handlebody  $H_\alpha$  (resp.  $H_\beta$ ).

Let  $\omega$  be an area form on  $\Sigma$ . We consider  $[0, 1] \times \Sigma$  with the stable Hamiltonian structure  $(dt, \omega)$ , where  $t$  is the  $[0, 1]$ -coordinate. The Hamiltonian vector field is  $\partial_t$ , and the 2-plane field  $\ker dt$  will be written as  $T\Sigma$ , at the risk of some confusion. If the map  $f : X \rightarrow \Sigma$  is understood, then we write  $T\Sigma_X$  or  $T\Sigma$  for the pullback bundle  $f^*T\Sigma$ , e.g.,  $T\Sigma_\Sigma$  or  $T\Sigma_{[0,1] \times \Sigma}$ .

Let  $\mathcal{S} = \mathcal{S}_{\alpha, \beta}$  be the set of  $k$ -tuples of chords  $\{[0, 1] \times \{y_1\}, \dots, [0, 1] \times \{y_k\}\}$  in  $[0, 1] \times \Sigma$ , where there exists a permutation  $\sigma \in \mathfrak{S}_k$  for which  $y_i \in \alpha_i \cap \beta_{\sigma(i)}$ ,  $i = 1, \dots, k$ . Here the chords  $[0, 1] \times \{y_i\}$  are orbits of the Hamiltonian vector field  $\partial_t$  which connect from  $\{0\} \times \beta$  to  $\{1\} \times \alpha$ .

*Terminology.* We will often write elements of  $\mathcal{S}$  as  $\mathbf{y} = \{y_1, \dots, y_k\}$  and refer to  $\mathbf{y}$  as a  *$k$ -tuple of intersection points*. Also, if  $l \leq k$ , then an  *$l$ -tuple of chords/intersection points*  $\mathbf{y} = \{y_1, \dots, y_l\}$  is a collection of points in  $\alpha \cap \beta$  where each  $\alpha_i$  is used at most once and each  $\beta_i$  is used at most once.

**4.2. Almost complex structures.** Consider the natural projection  $\pi_B : W \rightarrow B$ , where  $W = \mathbb{R} \times [0, 1] \times \Sigma$  and  $B = \mathbb{R} \times [0, 1]$ . We also write  $\pi_{\mathbb{R}}$ ,  $\pi_{[0,1] \times \Sigma}$  and  $\pi_\Sigma$  for the natural projections of  $W$  onto  $\mathbb{R}$ ,  $[0, 1] \times \Sigma$  and  $\Sigma$ . Let  $(s, t)$  be the coordinates on the base  $B = \mathbb{R} \times [0, 1]$ . We then define the symplectic form

$$\Omega = ds \wedge dt + \omega$$

on  $W$ . The submanifolds  $L_\alpha = \mathbb{R} \times \{1\} \times \alpha$  and  $L_\beta = \mathbb{R} \times \{0\} \times \beta$  are Lagrangian submanifolds of the symplectic manifold  $(W, \Omega)$ .

**Definition 4.2.1** ( $\Omega$ -admissibility). An almost complex structure  $J$  on  $W$  is  $\Omega$ -admissible (or simply *admissible*) if it satisfies the following:

- (1)  $J$  is  $s$ -invariant;
- (2)  $J(\partial_s) = \partial_t$  and  $J(T\Sigma) = T\Sigma$ ;
- (3)  $J$  is tamed by the symplectic form  $\Omega$ ;
- (4) there is a point  $z_i$  in each component of  $\Sigma - \alpha - \beta$  such that  $J$  is a product complex structure  $j_B \times j_\Sigma$  in a small neighborhood of  $\mathbb{R} \times [0, 1] \times \{z_i\}$  in  $W$ ; some  $z_i$  coincides with the basepoint  $z$ .

*Remark 4.2.2.* The fibers  $\{(s, t)\} \times \Sigma$  of an admissible  $J$  on  $W$  are holomorphic and the projection  $\pi_B$  is  $(J, j_B)$ -holomorphic, where  $j_B$  is the standard complex structure on  $B = \mathbb{R} \times [0, 1]$ .

We write  $\mathcal{J}_\Sigma$  for the space of  $C^\infty$ -smooth  $\Omega$ -admissible almost complex structures  $J$  on  $W$ .

**4.3. Holomorphic curves and moduli spaces.** Let  $(F, j)$  be a compact Riemann surface, possibly disconnected, with two sets of punctures  $\mathbf{q}^+ = \{q_1^+, \dots, q_k^+\}$  and  $\mathbf{q}^- = \{q_1^-, \dots, q_k^-\}$  on  $\partial F$ , such that (i) each component of  $F$  has nonempty boundary, (ii) on each boundary component there is at least one puncture from each of  $\mathbf{q}^+$  and  $\mathbf{q}^-$ , and (iii) the punctures on  $\mathbf{q}^+$  and  $\mathbf{q}^-$  alternate around each boundary component. We write  $\dot{F} = F - \mathbf{q}^+ - \mathbf{q}^-$  and  $\partial \dot{F} = \partial F - \mathbf{q}^+ - \mathbf{q}^-$ .

**Definition 4.3.1.** Let  $J \in \mathcal{J}_\Sigma$ . A *degree  $l \leq k$  multisection of  $(W, J)$*  is a holomorphic map

$$u : (\dot{F}, j) \rightarrow (W, J)$$

which is a degree  $l$  multisection of  $\pi_B : W \rightarrow B = \mathbb{R} \times [0, 1]$  and which *additionally satisfies the following*:

- (1)  $u(\partial \dot{F}) \subset L_\alpha \cup L_\beta$ ;
- (2) for each  $i \in \{1, \dots, k\}$ ,  $u^{-1}(L_{\alpha_i})$  (resp.  $u^{-1}(L_{\beta_i})$ ) consists of exactly one component of  $\partial \dot{F}$ , which we call  $\alpha_i^*$  (resp.  $\beta_i^*$ );
- (3)  $\lim_{w \rightarrow p_i} \pi_\mathbb{R} \circ u(w) = -\infty$  and  $\lim_{w \rightarrow q_i} \pi_\mathbb{R} \circ u(w) = +\infty$ ;
- (4) the energy of  $u$  (see Definition 4.3.2 below) is finite.

A *Heegaard Floer curve* (or *HF curve*) is a degree  $k$  multisection of  $(W, J)$ .

By the compactness theorem of [BEHWZ] (adapted to the Lagrangian case), a holomorphic curve  $u$  satisfying (1), (2) and (4) converges to cylinders over Reeb chords as  $s \rightarrow \pm\infty$ . By the work of Abbas [Ab], an HF curve  $u$  converges exponentially to cylinders over Reeb chords near the ends. Components of  $u$  may map to  $\mathbb{R} \times [0, 1] \times \{y_i\}$ ; such components will be called *trivial strips*.

**Definition 4.3.2.** The *energy* of  $u$  is the quantity

$$(4.3.1) \quad E(u) = \int_{\dot{F}} u^* \omega + \sup_{\phi \in \mathcal{C}} \int_{\dot{F}} u^* d(\phi(s) dt),$$

where  $\mathcal{C}$  is the set of nondecreasing smooth functions  $\phi : \mathbb{R} \rightarrow [0, 1]$ .

We now define some moduli spaces of HF curves with respect to  $J \in \mathcal{J}_\Sigma$ . Let  $\mathbf{y} = \{y_1, \dots, y_k\}$  and  $\mathbf{y}' = \{y'_1, \dots, y'_k\}$  be  $k$ -tuples of  $\alpha \cap \beta$ . Let  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  be

the moduli space of HF curves  $u$  which are asymptotic to  $\mathbb{R} \times [0, 1] \times \{y_i\}$  near  $q_i^+$  and to  $\mathbb{R} \times [0, 1] \times \{y'_i\}$  near  $q_i^-$ .<sup>4</sup> Such a curve  $u$  is said to be an *HF curve from  $\mathbf{y}$  to  $\mathbf{y}'$* . Also let  $\widehat{\mathcal{M}}_J(\mathbf{y}, \mathbf{y}') \subset \mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  be the subset consisting of curves  $u$  which additionally satisfy  $\pi_\Sigma \circ u(\dot{F}) \cap \{z\} = \emptyset$ .

Let  $\check{W} = [-1, 1] \times [0, 1] \times \Sigma$  be the compactification of  $W = \mathbb{R} \times [0, 1] \times \Sigma$ , obtained by attaching  $[0, 1] \times \Sigma$  at the positive and negative ends, and let  $\check{L}_\alpha = [-1, 1] \times \{1\} \times \alpha$  and  $\check{L}_\beta = [-1, 1] \times \{0\} \times \beta$  be the compactifications of  $L_\alpha$  and  $L_\beta$ . We then define  $Z_{\mathbf{y}, \mathbf{y}'} \subset \check{W}$  as the subset

$$(4.3.2) \quad Z_{\mathbf{y}, \mathbf{y}'} = \check{L}_\alpha \cup \check{L}_\beta \cup (\{1\} \times [0, 1] \times \mathbf{y}) \cup (\{-1\} \times [0, 1] \times \mathbf{y}').$$

Similarly define

$$(4.3.3) \quad Z_{\alpha, \beta} = \check{L}_\alpha \cup \check{L}_\beta \cup (\{-1, 1\} \times [0, 1] \times (\alpha \cap \beta)).$$

The exponential decay of HF curves in  $\mathbb{R} \times [0, 1] \times \Sigma$  implies that an HF curve  $u : \dot{F} \rightarrow W$  from  $\mathbf{y}$  to  $\mathbf{y}'$  can be compactified to a continuous map

$$\check{u} : (\check{F}, \partial \check{F}) \rightarrow (\check{W}, Z_{\mathbf{y}, \mathbf{y}'}).$$

Here  $\check{F}$  is obtained from  $\dot{F}$  by performing a real blow-up at its boundary punctures.

By some abuse of notation, let  $\pi_2(\mathbf{y}, \mathbf{y}')$  be the set of *homology* classes of continuous maps  $u : \dot{F} \rightarrow W$  which satisfy (1), (2) and (3) of Definition 4.3.1 and are positively asymptotic to  $[0, 1] \times \mathbf{y}$  and negatively asymptotic to  $[0, 1] \times \mathbf{y}'$ ; here two maps  $u_1$  and  $u_2$  are equivalent in  $\pi_2(\mathbf{y}, \mathbf{y}')$  if their compactifications  $\check{u}_1$  and  $\check{u}_2$  are homologous in  $H_2(\check{W}, Z_{\mathbf{y}, \mathbf{y}'})$ . To any HF curve from  $\mathbf{y}$  to  $\mathbf{y}'$  we can then associate a class in  $\pi_2(\mathbf{y}, \mathbf{y}')$ . If we consider moduli spaces of HF curves  $u$  in the homology class  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$ , we will write  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}', A)$  or  $\widehat{\mathcal{M}}_J(\mathbf{y}, \mathbf{y}', A)$ .

**4.4. The Fredholm index.** In this subsection and the next, we fix  $J \in \mathcal{J}_\Sigma$ .

In this subsection we discuss the Fredholm index of an HF curve  $u : \dot{F} \rightarrow W$ , which is the expected dimension of a neighborhood  $\mathcal{U}$  of  $u \in \mathcal{M}_J(\mathbf{y}, \mathbf{y}')$ , modulo reparametrizations of the domain. Since the curve  $u$  cannot be multiply-covered, the regularity of  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  is straightforward; see Lemma 4.7.2. The Fredholm index of  $u$  will be denoted by  $\text{ind}(u) = \text{ind}_{HF}(u)$ .

**4.4.1. The Fredholm index, first version.** We start with Lipshitz's formula [Li, Equation 5] for the Fredholm index of  $u$ :

$$(4.4.1) \quad \text{ind}(u) = -\chi(F) + k + \sum_{i=1}^k \mu(\alpha_i^*) - \sum_{i=1}^k \mu(\beta_i^*),$$

where  $k$  is the genus of  $\Sigma$ ,  $\chi(F) = \chi(\dot{F})$  is the Euler characteristic of  $F$  or  $\dot{F}$ , and  $\alpha_i^*$  and  $\beta_i^*$  are as in Definition 4.3.1.

We now define the Maslov indices  $\mu(\alpha_i^*)$  and  $\mu(\beta_i^*)$  which appear in Equation (4.4.1): Choose a trivialization  $\tau'_0$  of  $T\Sigma \simeq \mathbb{C}$  in a neighborhood of the points  $w \in \alpha \cap \beta$  so that  $\mathbb{R}$  corresponds to  $T_w \beta$  and  $i\mathbb{R}$  corresponds to  $T_w \alpha$ . Then let  $\tau_0$  be

<sup>4</sup>In [Li],  $W$  has the opposite orientation,  $\mathbf{y}$  is at  $-\infty$ , and  $\mathbf{y}'$  is at  $+\infty$ . The moduli spaces, however, are diffeomorphic.

a trivialization of  $u^*T\Sigma_W$  which coincides with the one already given near  $\mathbf{p}$  and  $\mathbf{q}$  by pulling back  $\tau'_0$ . Along each component of  $\partial\dot{F}$  — called  $\alpha_i^*$  or  $\beta_i^*$  depending on whether it is mapped to  $\alpha$  or to  $\beta$ , and oriented in the same way as  $\partial F$  — we have a loop of real lines in  $\mathbb{C}$ , given by the pullback of  $T\alpha$  or  $T\beta$ . The Maslov index  $\mu(\alpha_i^*)$  (resp.  $\mu(\beta_i^*)$ ) is twice the degree of the loop along  $\alpha_i^*$  (resp.  $\beta_i^*$ ) with respect to  $\tau_0$ .

**4.4.2. The Fredholm index, second version.** For the purposes of computing indices, we replace  $W = \mathbb{R} \times [0, 1] \times \Sigma$  with the compactification  $\check{W} = [-1, 1] \times [0, 1] \times \Sigma$  from Section 4.3.

Recall  $Z_{\alpha,\beta} \subset \check{W}$  which was given by Equation (4.3.3). We define a trivialization  $\tau$  of  $T\Sigma_{\check{W}}$  along  $Z_{\alpha,\beta} \subset \check{W}$  as follows: First choose a nonsingular tangent vector field along each component of  $\alpha$  and  $\beta$ . This induces a trivialization  $\tau'$  of  $T\Sigma_{[0,1] \times \Sigma}$  on  $(\{0\} \times \beta) \cup (\{1\} \times \alpha)$ . We then extend  $\tau'$  arbitrarily to  $[0, 1] \times (\alpha \cap \beta)$ . Finally we pull  $\tau'$  back to  $Z_{\alpha,\beta} \subset \check{W}$  using the projection  $\pi_{[0,1] \times \Sigma} : \check{W} \rightarrow [0, 1] \times \Sigma$  to obtain  $\tau$ .

Given an HF curve  $u : \dot{F} \rightarrow W$ , we define its *Maslov index*  $\mu_\tau(u)$  as follows: Let

$$\check{u} : (\check{F}, \partial\check{F}) \rightarrow (\check{W}, Z_{\alpha,\beta})$$

be the compactification of  $u$ . We then construct a (not necessarily oriented) real rank one subbundle  $\mathcal{L}$  of  $\check{u}^*T\Sigma$  on  $\partial\check{F}$ . The bundle  $\mathcal{L}$  is given by  $\check{u}^*T\alpha$  and  $\check{u}^*T\beta$  along  $\partial\check{F}$ . We extend  $\mathcal{L}$  to  $\partial\check{F} - \partial\dot{F}$  by rotating in the counterclockwise direction from  $\check{u}^*T\beta$  to  $\check{u}^*T\alpha$  by the minimum amount possible. (Assuming orthogonal intersections, this is a  $\frac{\pi}{2}$ -rotation.) Then  $\mu_\tau(u)$  is the sum of the Maslov indices of  $\mathcal{L}$  with respect to the trivialization  $\tau$ , where the sum is over all the connected components of  $\partial\check{F}$ .

**Lemma 4.4.1.** *If  $u : \dot{F} \rightarrow W$  is an HF curve, then*

$$(4.4.2) \quad \mu_\tau(u) + 2c_1(u^*T\Sigma, \tau) = \sum_{i=1}^k \mu(\alpha_i^*) - \sum_{i=1}^k \mu(\beta_i^*).$$

*Proof.* By standard Maslov index theory, we have

$$\mu_\tau(u) + 2c_1(u^*T\Sigma, \tau) = \mu_{\tau_0}(u) + 2c_1(u^*T\Sigma, \tau_0),$$

where  $\tau_0$  denotes the trivialization of  $u^*T\Sigma$  from Section 4.4.1. We immediately obtain  $c_1(u^*T\Sigma, \tau_0) = 0$  since  $\tau_0$  is a trivialization on all of  $\dot{F}$ . Hence it suffices to prove that:

$$(4.4.3) \quad \mu_{\tau_0}(u) = \sum_{i=1}^k \mu(\alpha_i^*) - \sum_{i=1}^k \mu(\beta_i^*).$$

The difference between the two sides of Equation (4.4.3) is the total amount of rotation of the real lines introduced at  $\alpha \cap \beta$  in the definition of  $\mu_{\tau_0}$ : if we go from  $\beta$  to  $\alpha$  we rotate by  $\frac{\pi}{2}$ , while if we go from  $\alpha$  to  $\beta$  we rotate by  $-\frac{\pi}{2}$ ; hence the total amount of rotation is 0.  $\square$

We can now rephrase the Fredholm index as follows:

$$(4.4.4) \quad \text{ind}(u) = -\chi(F) + k + \mu_\tau(u) + 2c_1(u^*T\Sigma, \tau).$$

*Remark 4.4.2.* Since the Maslov index of  $\mathcal{L}$  with respect to  $\tau$  is an integer along each chord  $[0, 1] \times \{y_i\}$ , it makes sense to write  $\mu_\tau(y_i) \in \mathbb{Z}$ . If we let

$$\mu_\tau(\mathbf{y}) = \sum_{i=1}^k \mu_\tau(y_i),$$

then  $\mu_\tau(u) = \mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}')$ . In particular,  $\mu_\tau(u)$  only depends on  $\mathbf{y}$ ,  $\mathbf{y}'$ , and the choice of  $\tau$ .

**4.5. The ECH-type index.** In this subsection we define an ECH-type index  $I_{HF}(u)$  and prove an index inequality which is analogous to the ECH index inequality of [Hu1].

**4.5.1.  $\tau$ -trivial representatives.** Let  $\tau$  be a trivialization of  $T\Sigma_{\tilde{W}}$  along  $Z_{\alpha, \beta}$  as defined in Section 4.4.2. We will define the notions of a *representative* and a  $\tau$ -*trivial representative* of a homology class  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$ , where  $\mathbf{y} = \{y_1, \dots, y_k\}$  and  $\mathbf{y}' = \{y'_1, \dots, y'_k\}$  are  $k$ -tuples in  $\mathcal{S}_{\alpha, \beta}$ .

**Definition 4.5.1.** An oriented immersed compact surface

$$\check{C} \subset \check{W} = [-1, 1] \times [0, 1] \times \Sigma$$

in the homology class  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$  is an *immersed representative* of  $A$  if:

- (1)  $\check{C}$  is positively transverse to the fibers  $\{(s, t)\} \times \Sigma$  along all of  $\partial\check{C}$ .
- (2)  $(\tilde{\pi}_{[0, 1] \times \Sigma})|_{\check{C}}$  is an embedding near  $\partial\check{C} \cap (\{-1, 1\} \times [0, 1] \times \Sigma)$ , where  $\tilde{\pi}_{[0, 1] \times \Sigma}$  is the projection  $\check{W} \rightarrow [0, 1] \times \Sigma$ .

If  $\check{C}$  is embedded in addition, then  $\check{C}$  is a *representative* of  $A$ .

**Definition 4.5.2** ( $\tau$ -trivial representative). A representative  $\check{C}$  of  $A$  is  $\tau$ -*trivial* if, for all sufficiently small  $\varepsilon > 0$ ,  $\check{C} \cap \{s = \pm(1 - \varepsilon)\}$  is the union of single-stranded braids  $\zeta_i^\pm$ ,  $i = 1, \dots, k$ , where  $\zeta_i^+$  (resp.  $\zeta_i^-$ ) lies in a tubular neighborhood of  $[0, 1] \times \{y_i\}$  (resp.  $[0, 1] \times \{y'_i\}$ ), is disjoint from  $[0, 1] \times \{y_i\}$  (resp.  $[0, 1] \times \{y'_i\}$ ), and induces a framing which agrees with  $\tau$  along  $[0, 1] \times \{y_i\}$  (resp.  $[0, 1] \times \{y'_i\}$ ).

Let  $A$  be a homology class in  $\pi_2(\mathbf{y}, \mathbf{y}')$ . Then we define

$$n_{z_j}(A) = \langle A, [-1, 1] \times [0, 1] \times \{z_j\} \rangle,$$

where  $z_j \in \Sigma - \alpha - \beta$  are given in Definition 4.2.1 and  $\langle \cdot, \cdot \rangle$  is the signed intersection number. We say that  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$  is *positive* if  $n_{z_j}(A) \geq 0$  is nonnegative for all  $z_j$ .

**Lemma 4.5.3.** A positive  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$  admits a  $\tau$ -trivial representative  $\check{C}$ .

*Proof.* Let  $A$  be a positive homology class in  $\pi_2(\mathbf{y}, \mathbf{y}')$ . Then we can glue closures of connected components of  $\Sigma - \alpha - \beta$  with multiplicity  $n_{z_j}(A)$  as in Rasmussen [Ra, Lemma 9.3] (also see [Li, Lemma 4.1]) to construct a continuous map  $u_2 : F \rightarrow \Sigma$  which is smooth on  $\dot{F}$  and satisfies the following:

- $u_2(q_i^+) = y_i$  and  $u_2(q_i^-) = y'_i$ ;
- each component of  $\partial\tilde{F}$  is mapped to some  $\alpha_i$  or  $\beta_i$  so that each  $\alpha_i, \beta_i$ ,  $i = 1, \dots, k$ , is used exactly once;
- $u_2|_{\partial F}$  switches from  $\beta$  to  $\alpha$  (resp.  $\alpha$  to  $\beta$ ) in a neighborhood of  $y_i$  (resp.  $y'_i$ ) as we move in the direction given by the orientation of  $\partial F$ .

The map  $u_2$  can be pulled back to  $\tilde{u}_2 : \tilde{F} \rightarrow \Sigma$ , where  $\tilde{F}$  is the real blow-up of  $F$  given in Section 4.3. Now take a branched cover  $\tilde{u}_1 : \tilde{F} \rightarrow [-1, 1] \times [0, 1]$  with interior branch points, and form  $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) : \tilde{F} \rightarrow \tilde{W}$ . Condition (1) of Definition 4.5.1 is immediately satisfied. We can resolve all the (interior) singularities to make  $\tilde{u}$  embedded. It is a local exercise to modify  $\tilde{u}$  in a neighborhood of  $\partial\tilde{F} - \partial\tilde{F}$  so that  $\tilde{u}$  becomes  $\tau$ -trivial.  $\square$

**Lemma 4.5.4.** *If  $u : \dot{F} \rightarrow W$  is an HF curve and  $\check{C}$  is the image of the compactification  $\tilde{u} : \tilde{F} \rightarrow \tilde{W}$ , then the following hold:*

- (1)  $\pi_B \circ u : \dot{F} \rightarrow B$  has no branch points along  $\partial B$ .
- (2)  $\tilde{u}$  is positively transverse to the fibers  $\{(s, t)\} \times \Sigma$  along all of  $\partial\tilde{F}$  and  $\tilde{\pi}_{[0,1] \times \Sigma}|_{\check{C}}$  is an embedding near  $\partial\check{C}$ .

In other words,  $\check{C}$  satisfies all the conditions of a  $\tau$ -trivial representative for some  $\tau$ , with the exception of the embeddedness of  $\check{C}$ .

*Proof.* (1) follows from the fact that  $\pi_B \circ u$  is a  $k$ -fold branched cover of  $B$ . Let  $\mathbb{H} = \{\text{Im}(z) \geq 0\}$  be the upper half-plane and  $U \subset \mathbb{H}$  be an open subset which contains 0. If  $f$  is a holomorphic map  $U \rightarrow \mathbb{R} \times [0, 1]$  which maps 0 to  $(0, 0)$  and  $U \cap \partial\mathbb{H}$  to  $\mathbb{R} \times \{0\}$ , then it can be extended to a holomorphic map  $f : U \cup \overline{U} \rightarrow \mathbb{R} \times [-1, 1]$  by Schwarz reflection, where  $\overline{U} = \{\bar{z} \mid z \in U\}$ . If  $df(0) = 0$ , then  $f$  is locally a composition of  $z \mapsto z^l$  for some integer  $l > 1$  and a biholomorphism. This contradicts the requirement that  $f(\mathbb{H})$  stay on one side of  $\mathbb{R} \times \{0\}$ .

(2) follows from (1), together with the asymptotics of  $u$  as  $s \rightarrow \pm\infty$ .  $\square$

**4.5.2. The relative intersection form.** We now define the relative intersection form  $Q_\tau(A)$ , which is analogous to the relative intersection form which appears in the definition of the ECH index  $I_{ECH}$ , but is easier.

**Definition 4.5.5** (Relative intersection form  $Q_\tau(A)$ ). Let  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$  be a positive homology class and let  $\check{C}$  be a  $\tau$ -trivial representative of  $A$ . Let  $\psi$  be a section of the normal bundle  $\nu$  to  $\check{C}$  such that  $\psi|_{\partial\check{C}} = J\tau$ , and let  $\check{C}'$  be a pushoff of  $\check{C}$  in the direction of  $\psi$ . Then the *relative intersection form*  $Q_\tau(A)$  is given by:

$$Q_\tau(A) = \langle \check{C}, \check{C}' \rangle$$

where  $\langle \cdot, \cdot \rangle$  denotes the algebraic count of intersection points.

Note that, since a representative  $\check{C}$  is positively transverse to the fibers  $\{(s, t)\} \times \Sigma$  along all of  $\partial\check{C}$ , we may take the normal bundle  $\nu$  to  $\check{C}$  to satisfy  $\nu|_{\partial\check{C}} = T\Sigma|_{\partial\check{C}}$ . Also, since  $J$  is  $\Omega$ -admissible, it takes  $T\Sigma$  to itself. Hence  $(\tau, J\tau)$  is a trivialization of  $\nu|_{\partial\check{C}}$ . Although  $\tau$  and  $J\tau$  are homotopic, we will often use  $J\tau$  due to its appearance in the definition of  $Q_\tau(A)$ .

**4.5.3. Properties of the relative intersection form.** Let  $\tau'$  and  $\tau$  be trivializations of  $T\Sigma_{\tilde{W}}$  along  $Z_{\alpha,\beta}$  which differ only on  $\{\pm 1\} \times [0, 1] \times (\alpha \cap \beta)$ . Let  $\deg(\tau, \tau', y_i)$  (resp.  $\deg(\tau, \tau', y'_i)$ ) be the degree of  $\tau$  with respect to the trivialization  $\tau'$  along  $[0, 1] \times \{y_i\}$  (resp.  $[0, 1] \times \{y'_i\}$ ), oriented in the  $\partial_t$ -direction. We then have the following:

**Lemma 4.5.6** (Change of trivialization). *If  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$  is positive, then*

$$(4.5.1) \quad Q_\tau(A) - Q_{\tau'}(A) = \sum_{i=1}^k \deg(\tau, \tau', y_i) - \sum_{i=1}^k \deg(\tau, \tau', y'_i).$$

*Proof.* Let  $\check{C}_{\tau'}$  be a  $\tau'$ -trivial representative of  $A$ . Let  $\varepsilon > 0$  be small and let

$$\check{C}_{\tau',0} = \check{C}_{\tau'} \cap ([-1 + \varepsilon, 1 - \varepsilon] \times [0, 1] \times \Sigma).$$

We can extend  $\check{C}_{\tau',0}$  to  $\check{C}_\tau$  on  $\tilde{W}$  by gluing disks  $D_i, D'_i, i = 1, \dots, k$ , corresponding to  $y_i, y'_i$ , so that  $\check{C}_\tau$  becomes  $\tau$ -trivial.

Let  $\psi$  be a section of the normal bundle to  $\check{C}_\tau$  such that  $\psi|_{\partial\check{C}_\tau} = J\tau$  and  $\psi|_{\partial\check{C}_{\tau',0}} = J\tau'$ . (Here we are assuming that  $\tau'$  has been extended to a neighborhood of the  $[0, 1] \times \{y_i\}$ .) Then

$$Q_\tau(A) - Q_{\tau'}(A) = \sum_{i=1}^k \#(\psi|_{D_i})^{-1}(0) + \sum_{i=1}^k \#(\psi|_{D'_i})^{-1}(0),$$

where  $\#$  is a signed count. A local calculation gives

$$\#(\psi|_{D_i})^{-1}(0) = \deg(\tau, \tau', y_i), \quad \#(\psi|_{D'_i})^{-1}(0) = -\deg(\tau, \tau', y'_i),$$

which proves the lemma.  $\square$

The following is immediate from the definition of  $Q_\tau(A)$ .

**Lemma 4.5.7** (Additivity). *If  $A_1 \in \pi_2(\mathbf{y}, \mathbf{y}')$  and  $A_2 \in \pi_2(\mathbf{y}', \mathbf{y}'')$  are positive, then*

$$Q_\tau(A_1 \# A_2) = Q_\tau(A_1) + Q_\tau(A_2),$$

where  $A_1 \# A_2 \in \pi_2(\mathbf{y}, \mathbf{y}'')$  is obtained from stacking two copies of  $\tilde{W}$ .

**4.5.4. The relative adjunction formula.** In this subsection we prove the relative adjunction formula (Lemma 4.5.9).

Let  $\delta(\check{C})$  be the signed count of singularities of  $\check{C}$  in its interior. In particular, if  $\check{C}$  is immersed, then  $\delta(\check{C})$  is the signed count of transverse double points of  $\check{C}$ . We will also use the notation  $\delta(u)$  or  $\delta(\check{u})$ .

**Lemma 4.5.8.** *If  $\check{C}$  is an immersed representative of  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$ , then*

$$c_1(\nu, J\tau) = Q_\tau(A) - 2\delta(\check{C}).$$

*Proof.* Let us first assume that  $\check{C}$  is embedded. If  $\check{C}$  is  $\tau$ -trivial, then there is a section  $\psi$  of the normal bundle to  $\check{C}$  such that  $\psi|_{\partial\check{C}} = J\tau$ , and

$$Q_\tau(A) = \#\psi^{-1}(0) = c_1(\nu, J\tau).$$

Next let  $\tau'$  and  $\tau$  be trivializations of  $T\Sigma_{\check{W}}$  along  $Z_{\alpha,\beta}$ , which differ only on  $\{\pm 1\} \times [0, 1] \times (\alpha \cap \beta)$ . By Equation (4.5.1), together with an analogous equation for  $c_1(\nu, J\tau) - c_1(\nu, J\tau')$ , we have

$$Q_\tau(A) - Q_{\tau'}(A) = c_1(\nu, J\tau) - c_1(\nu, J\tau'),$$

which proves the lemma for embedded  $\check{C}$ .

Suppose now that  $\check{C}$  has a single positive transverse double point  $d$ . (The case of a negative double point is similar.) We resolve the intersection in the following way: Let  $B \subset \check{W}$  be a small ball centered at  $d$ . Then  $\check{C} \cap \partial B$  is a Hopf link, and  $\check{C} \cap B$  is the union of two slice disks for the components which intersect at  $d$ . We can construct a new surface  $\check{C}_{sm}$  by replacing the two disks with a Hopf band connecting the two components of the Hopf link. By definition, we have  $Q_\tau(A) = \langle \check{C}_{sm}, \check{C}'_{sm} \rangle$ . On the other hand, if  $\nu_{sm}$  is the normal bundle to  $\check{C}_{sm}$ , then

$$c_1(\nu_{sm}, J\tau) = c_1(\nu, J\tau) + 2.$$

This can be seen easily by embedding  $B$  into  $S^2 \times S^2$  and using the properties of the intersection product for closed 4-manifolds.

In general,

$$c_1(\nu, J\tau) + 2\delta(\check{C}) = c_1(\nu_{sm}, J\tau) = \langle \check{C}_{sm}, \check{C}'_{sm} \rangle = Q_\tau(A),$$

and the lemma follows.  $\square$

We can now state and prove the relative adjunction formula:

**Lemma 4.5.9** (Relative adjunction formula). *If  $u : \dot{F} \rightarrow W$  is an HF curve in the homology class  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$ , then*

$$(4.5.2) \quad \begin{aligned} c_1(\check{u}^*T\Sigma, \tau) &= c_1(T\check{F}, \partial_t) + Q_\tau(A) - 2\delta(u) \\ &= \chi(F) - k + Q_\tau(A) - 2\delta(u). \end{aligned}$$

Here  $\delta(u)$  is a nonnegative integer which equals 0 if and only if  $u$  is an embedding, and  $\partial_t$  is the pullback to  $\partial\check{F}$  of the trivialization  $\partial_t$  on  $[-1, 1] \times [0, 1]$ .

*Proof.* By [M1, MW], there exists a modification  $v : \dot{F} \rightarrow W$  of  $u : \dot{F} \rightarrow W$  in a neighborhood of its finitely many singular points so that  $v$  is symplectic with only transverse double points. Since the modification is purely local and is away from  $\partial\dot{F}$ , it follows that  $\check{u}$  and  $\check{v}$  belong to the same homology class in  $\pi_2(\mathbf{y}, \mathbf{y}')$  and  $c_1(\check{u}^*T\Sigma, \tau) = c_1(\check{v}^*T\Sigma, \tau)$ . Hence Equation (4.5.2) for  $u$  is equivalent to Equation (4.5.2) for  $v$ , and we may assume without loss of generality that  $u$  is immersed with positive transverse double points.

The vector field  $\partial_t$  is a global trivialization of the complex line bundle  $T([-1, 1] \times [0, 1])$  over  $\check{W} = [-1, 1] \times [0, 1] \times \Sigma$ . Hence

$$c_1(\check{u}^*T\check{W}, (\tau, \partial_t)) = c_1(\check{u}^*T\Sigma, \tau).$$

On the other hand,

$$c_1(\check{u}^*T\check{W}, (\tau, \partial_t)) = c_1(T\check{F}, \partial_t) + c_1(\nu, J\tau).$$



The first line of the relative adjunction formula now follows from Lemma 4.5.8. The equivalence of the first and second lines is a consequence of Claim 4.5.10, proved below.  $\square$

**Claim 4.5.10.**  $c_1(T\check{F}, \partial_t) = \chi(F) - k$ .

*Proof.* Let  $\tau_{\partial F}$  be the trivialization of  $TF|_{\partial F}$  which is given by an oriented non-singular vector field tangent to  $\partial F$ . We then have

$$c_1(T\check{F}, \partial_t) = \chi(F) + \deg(\partial_t, \tau_{\partial F}),$$

where  $\deg(\partial_t, \tau_{\partial F})$  is the degree of  $\partial_t$  with respect to  $\tau_{\partial F}$ . By an easy direct calculation we obtain  $\deg(\partial_t, \tau_{\partial F}) = -k$ .  $\square$

**4.5.5. The index  $I_{HF}$  and the index inequality.** We are now ready to define the ECH-type index  $I_{HF}$  and prove the ECH-type index inequality (Theorem 4.5.13).

**Definition 4.5.11** (ECH-type index). Let  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$  be a positive homology class. Then the *ECH-type index*  $I_{HF}$  of  $A$  is given as follows:

$$(4.5.3) \quad I_{HF}(A) = c_1(T\Sigma|_A, \tau) + Q_\tau(A) + \mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}').$$

We observe that  $I_{HF}(A)$  does not depend on the choice of  $\tau$ : Suppose  $\tau$  and  $\tau'$  differ only at  $y_i$  and  $\deg(\tau, \tau', y_i) = 1$ . Then we compute (i)  $Q_\tau(A) = Q_{\tau'}(A) + 1$  by Lemma 4.5.6, (ii)  $\mu_\tau(\mathbf{y}) = \mu_{\tau'}(\mathbf{y}) - 2$  and (iii)  $c_1(T\Sigma|_A, \tau) = c_1(T\Sigma|_A, \tau') + 1$ .

The index  $I_{HF}$  satisfies the following additivity property:

**Lemma 4.5.12** (Additivity of  $I_{HF}$ ). *If  $A_1 \in \pi_2(\mathbf{y}, \mathbf{y}')$  and  $A_2 \in \pi_2(\mathbf{y}', \mathbf{y}'')$  are positive, then*

$$I_{HF}(A_1 \# A_2) = I_{HF}(A_1) + I_{HF}(A_2).$$

*Proof.* Each of the terms  $c_1(T\Sigma|_A, \tau)$ ,  $Q_\tau(A)$ , and  $\mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}')$  in the definition of  $I_{HF}(A)$  is additive under stacking; see Lemma 4.5.7 for the additivity of  $Q_\tau(A)$ .  $\square$

The following index inequality is analogous to (but much easier than) the ECH index inequality, due to Hutchings [Hu1, Theorem 1.7]. We remark that  $u$  is required to be simply-covered in the statement of the usual ECH index inequality. This is automatically satisfied for HF-curves.

**Theorem 4.5.13** (ECH-type index inequality). *Let  $u : \dot{F} \rightarrow W$  be an HF curve in the class  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$ . Then*

$$(4.5.4) \quad \text{ind}(u) + 2\delta(u) = I_{HF}(A),$$

where  $\delta(u) \geq 0$  is an integer count of the singularities. Hence

$$(4.5.5) \quad \text{ind}(u) \leq I_{HF}(A),$$

with equality if and only if  $u$  is an embedding.

*Proof.* We calculate:

$$\begin{aligned} \text{ind}(u) &= -\chi(F) + k + \mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}') + 2c_1(\check{u}^*T\Sigma, \tau) \\ &= c_1(\check{u}^*T\Sigma, \tau) + Q_\tau(A) + \mu_\tau(\mathbf{y}) - \mu_\tau(\mathbf{y}') - 2\delta(u). \end{aligned}$$

The first line is Equation (4.4.4). The equivalence of the first and second lines follows from the relative adjunction formula. Hence

$$\text{ind}(u) + 2\delta(u) = I_{HF}(A).$$

The index inequality (4.5.5) follows immediately.  $\square$

**4.6. Compactness.** We now discuss the requisite compactness issues. The key notion is that of *weak admissibility*, which is analogous to the vanishing of the flux in the PFH situation (see Section 3.3). Let  $(\Sigma, \alpha, \beta, z)$  be a *weakly admissible Heegaard diagram*, i.e., for every  $\text{Spin}^c$ -structure  $\mathfrak{s}$  and nontrivial periodic domain  $\mathcal{Q}$  which satisfies  $\langle c_1(\mathfrak{s}), \mathcal{Q} \rangle = 0$ , there exist  $j_1$  and  $j_2$  for which  $n_{z_{j_1}}(\mathcal{Q}) > 0$  and  $n_{z_{j_2}}(\mathcal{Q}) < 0$ . Equivalently, by [OSz1, Lemma 4.12],  $(\Sigma, \alpha, \beta, z)$  is weakly admissible if and only if there is an area form  $\omega$  on  $\Sigma$  such that each periodic domain has total signed  $\omega$ -area zero.

Let  $N > 0$  be a fixed constant. We consider the subset  $\pi_2^N(\mathbf{y}, \mathbf{y}')$  consisting of homology classes of  $\pi_2(\mathbf{y}, \mathbf{y}')$  which intersect  $[-1, 1] \times [0, 1] \times \{z\}$  at most  $N$  times. (This is sufficient for  $\widehat{CF}$  and  $CF^+$ , defined in the next subsection.) The difference of two homology classes  $A_1, A_2 \in \pi_2^0(\mathbf{y}, \mathbf{y}')$  is a periodic domain  $\mathcal{Q}$  and has zero  $\omega$ -area. This implies that the  $\omega$ -areas of any two  $A_1, A_2 \in \pi_2^N(\mathbf{y}, \mathbf{y}')$  differ by  $i \cdot \omega(\Sigma)$  where  $0 \leq i \leq N$ . Let  $\phi_1, \dots, \phi_r$  be the connected components of  $\Sigma - \alpha - \beta$ . If  $A$  is represented by a holomorphic curve, then the projection of  $A$  to  $\Sigma$  can be written as  $\sum_j n_j(A) \phi_j$  with  $n_j(A) \geq 0$ . Since each  $\phi_i$  has finite area, there must only be a finite number of homology classes  $A \in \pi_2^N(\mathbf{y}, \mathbf{y}')$  for which the moduli space  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}', A)$  is nonempty.

We now prove the existence of a compactification of  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}', A)/\mathbb{R}$ . It suffices to show that if  $u : \dot{F} \rightarrow W$  is an element of  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}', A)$ , then the genus of  $\dot{F}$  is bounded as long as  $A$  is fixed. This will be carried out in Lemma 4.6.1. Once we have a genus bound, the SFT compactness theorem from [BEHWZ] can be applied to give a compactification of  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}', A)/\mathbb{R}$ .

**Lemma 4.6.1.** *There is an upper bound on the genus of a holomorphic curve  $u : \dot{F} \rightarrow W$  in a fixed homology class  $A \in \pi_2(\mathbf{y}, \mathbf{y}')$ .*

*Proof.* The proof is analogous to the proof in the PFH case. In view of the relative adjunction formula (Lemma 4.5.9) and the nonnegativity of  $\delta(u)$ , we have

$$(4.6.1) \quad \chi(\dot{F}) \geq c_1(\check{u}^*T\Sigma, \tau) + k - Q_\tau(A).$$

The lemma follows by observing that the terms on the right-hand side depend only on the homology class  $A$ .  $\square$

#### 4.7. Transversality.

**Definition 4.7.1.** An almost complex structure  $J \in \mathcal{J}_\Sigma$  is *regular* if the moduli spaces  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  for all  $\mathbf{y}, \mathbf{y}' \in \mathcal{S} = \mathcal{S}_{\alpha, \beta}$  are transversely cut out.

Note that if  $u$  is an HF curve, then it does not have any closed irreducible components by definition. In particular,  $u$  cannot have any fibers  $\{(s, t)\} \times \Sigma$  as irreducible components.

We write  $\mathcal{J}_\Sigma^{reg} \subset \mathcal{J}_\Sigma$  for the subset of regular almost complex structures  $J$ . For  $J \in \mathcal{J}_\Sigma^{reg}$ , the dimension of  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  near  $u$  is equal to the *Fredholm index*  $\text{ind}(u)$ . The moduli space  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  carries a natural  $\mathbb{R}$ -action given by translations in the  $s$ -direction, and the quotient  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')/\mathbb{R}$  is a manifold.

**Lemma 4.7.2.** A generic  $J \in \mathcal{J}_\Sigma$  is regular.

*Proof.* This follows from [Li, Proposition 3.8], by noting that an HF curve  $u$  does not have any fibers as irreducible components. Lemma 4.7.2 can also be proved in the same way as in [Hu1, Lemma 9.12(b)]. Note that the transversality theory is relatively straightforward because HF curves are never multiply-covered, i.e., all the moduli spaces  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  consist of simple curves.  $\square$

We will use the notation  $\mathcal{M}_J^{I=r}(\mathbf{y}, \mathbf{y}')$  to denote the moduli space of HF curves from  $\mathbf{y}$  to  $\mathbf{y}'$  with ECH index  $I = r$ .

**Corollary 4.7.3** (Corollary of Theorem 4.5.13). *If  $I_{HF}(A) = 0, 1$  and  $J \in \mathcal{J}_\Sigma^{reg}$ , then every HF curve  $u$  in the class  $A$  satisfies  $\text{ind}(u) = I_{HF}(A)$  and is therefore embedded.*

*Proof.* This follows from Equation (4.5.4) by observing that the term  $2\delta(u)$  is even and nonnegative and that  $\text{ind}(u) \geq 0$  since  $J$  is regular and  $u$  is not multiply-covered.  $\square$

**4.8. Definition of the Heegaard Floer homology groups.** Let  $(\Sigma, \alpha, \beta, z)$  be a weakly admissible Heegaard diagram and let  $J \in \mathcal{J}_\Sigma^{reg}$ . We define the Heegaard Floer chain complexes

$$(\widehat{CF}(\Sigma, \alpha, \beta, z, J), \widehat{\partial}) \text{ and } (CF^+(\Sigma, \alpha, \beta, z, J), \partial^+),$$

whose corresponding homology groups are

$$\widehat{HF}(\Sigma, \alpha, \beta, z, J) \text{ and } HF^+(\Sigma, \alpha, \beta, z, J).$$

The hat group  $\widehat{CF}(\Sigma, \alpha, \beta, z, J)$  is the  $\mathbb{F}$ -vector space generated by  $\mathcal{S} = \mathcal{S}_{\alpha, \beta}$  and the plus group  $CF^+(\Sigma, \alpha, \beta, z, J)$  is the  $\mathbb{F}$ -vector space generated by  $\mathcal{S} \times \mathbb{Z}^{\geq 0}$ . Elements of  $\mathcal{S}$  will be written as  $\mathbf{y}$  and elements of  $\mathcal{S} \times \mathbb{Z}^{\geq 0}$  will be written as  $[\mathbf{y}, i]$ .

We now define the differentials  $\widehat{\partial}$  and  $\partial^+$ . The differential  $\widehat{\partial}$  is given by

$$\widehat{\partial}\mathbf{y} = \sum_{\mathbf{y}' \in \mathcal{S}} \langle \widehat{\partial}\mathbf{y}, \mathbf{y}' \rangle \cdot \mathbf{y}',$$

where  $\langle \widehat{\partial} \mathbf{y}, \mathbf{y}' \rangle$  is the count of  $\widehat{\mathcal{M}}_J^{I=1}(\mathbf{y}, \mathbf{y}')/\mathbb{R}$ . The differential  $\partial^+$  is given by

$$\partial^+([y, i]) = \sum_{[\mathbf{y}', j] \in \mathcal{S} \times \mathbb{Z}^{\geq 0}} \langle \partial^+([y, i]), [\mathbf{y}', j] \rangle \cdot [\mathbf{y}', j],$$

where  $\langle \partial^+([y, i]), [\mathbf{y}', j] \rangle$  is the count of  $\mathcal{M}_J^{I=1}(\mathbf{y}, \mathbf{y}')/\mathbb{R}$  whose representatives have intersection number  $i - j$  with  $\mathbb{R} \times [0, 1] \times \{z\}$ . By Theorem 4.5.13, the count of  $I_{HF}(u) = 1$  curves is equivalent to the count of embedded  $\text{ind}(u) = 1$  curves. Hence our definition is the same as that of Lipshitz.

The differentials  $\widehat{\partial}$  and  $\partial^+$  indeed satisfy  $\widehat{\partial}^2 = 0$  and  $(\partial^+)^2 = 0$  by [Li]. A tricky issue which arises for  $\partial^+$  (but not for  $\widehat{\partial}$ ) is that an element  $u$  of the boundary of  $\mathcal{M}_J^{I=2}(\mathbf{y}, \mathbf{y}')/\mathbb{R}$  might a priori have a fiber  $\{(s, t)\} \times \Sigma$  as an irreducible component. (In that case,  $u$  consists of one copy of  $\Sigma$ , together with  $k$  trivial strips.) This possibility is eliminated in [Li, Lemma 8.2].

Both  $\widehat{HF}(\Sigma, \alpha, \beta, z, J)$  and  $HF^+(\Sigma, \alpha, \beta, z, J)$  are independent of the choices and can be written as  $\widehat{HF}(M)$  and  $HF^+(M)$ . In this paper, we are interested in  $\widehat{HF}(-M)$ , where  $-M$  is the manifold  $M$  with the opposite orientation. The group  $\widehat{HF}(-M)$  is the homology of the chain complex  $\widehat{CF}(\Sigma, \beta, \alpha, z, J)$ , i.e., the complex for which the roles of the  $\alpha$ - and  $\beta$ -curves are exchanged.

**4.9. Restricting the complex to a page.** In this subsection we describe a pointed Heegaard diagram for  $-M$  which is adapted to an open book and which has the property that  $\widehat{HF}(-M)$  can be computed from a single page. The Heegaard diagram constructed here is a slight modification of the Heegaard diagram constructed by Honda, Kazez and Matić in [HKM1].

Let  $S$  be a bordered surface of genus  $g$  and connected boundary, and let  $(S, h)$  be an open book decomposition of  $M$  with binding  $K$ , i.e., there is a homeomorphism

$$((S \times [0, 1])/\sim, (\partial S \times [0, 1])/\sim) \simeq (M, K),$$

where  $(x, 1) \sim (h(x), 0)$  for  $x \in S$  and  $(y, t) \sim (y, t')$  for  $y \in \partial S$  and  $t, t' \in [0, 1]$ .

**4.9.1. A Heegaard diagram compatible with  $(S, h)$ .** We define a pointed Heegaard diagram  $(\Sigma, \beta, \alpha, z)$  for  $-M$  which is compatible with  $(S, h)$ . (In particular, this means that  $L_\beta = \mathbb{R} \times \{1\} \times \beta$  and  $L_\alpha = \mathbb{R} \times \{0\} \times \alpha$ .)

The open book decomposition  $(S, h)$  gives a natural Heegaard decomposition of  $M$  into two handlebodies  $H_1 = (S \times [0, \frac{1}{2}])/\sim$  and  $H_2 = (S \times [\frac{1}{2}, 1])/\sim$ . The Heegaard surface  $\Sigma$  is  $S_{1/2} \cup -S_0$  and has genus  $2g$ . Here we abbreviate  $S \times \{t\}$  by  $S_t$ .

A *basis* of  $S$  is a collection of properly embedded pairwise disjoint arcs  $\mathbf{a} = \{a_1, \dots, a_{2g}\}$  of  $S$  such that  $S - \mathbf{a}$  is a connected  $8g$ -gon. Given a basis  $\mathbf{a}$  of  $S$ , there is a natural collection of compression curves  $\alpha_i = \partial(a_i \times [0, \frac{1}{2}])$  for  $H_1$ . We write  $\alpha_i = a_i^\dagger \cup a_i$ , where the presence of  $\dagger$  indicates a copy of an arc in  $S_{1/2}$  and the absence indicates a copy of an arc in  $S_0$ . Recall the monodromy  $h$  maps  $(y, \theta) \mapsto (y, \theta - y)$  near  $\partial S$ . We then construct a collection of compression curves  $\beta_i = b_i^\dagger \cup h(a_i)$  for  $H_2$ , where  $b_i$  is the simplest arc (= fewest number

of intersections with the  $a_j$ ) in  $S_{1/2}$  which is parallel to  $a_i$  and extends  $h(a_i)$  to smooth curve in  $\Sigma$ . See Figure 1.

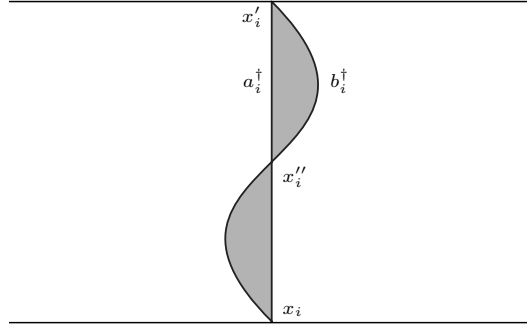


FIGURE 1. A portion of  $S_{1/2}$ . The shaded regions are the disks  $D_i$  and  $D'_i$ .

The arcs  $a_i$  and  $h(a_i)$  intersect at their endpoints  $x_i$  and  $x'_i$  by the definition of  $h$  near  $\partial S$ , and the arcs  $a_i$  and  $b_i$  intersect at a unique point  $x''_i$  in  $\text{int}(S_{1/2})$ . This means that all the intersection points of  $\alpha_i \cap \beta_j$  lie in  $S_0$ , except for one intersection point  $x''_i$  of  $\alpha_i \cap \beta_i$  for each  $i$ . We then place the basepoint  $z$  on the binding, away from all the intersection points  $x_i, x'_i$ . The regions of  $\Sigma - \alpha - \beta$  which nontrivially intersect  $S_{1/2}$  are the following:

- the “forbidden region” containing the basepoint  $z$ ;
- for each  $i = 1, \dots, 2g$ , a bigon  $D_i$  from  $x''_i$  to  $x_i$  and a bigon  $D'_i$  from  $x''_i$  to  $x'_i$ .

By the placement of the basepoint  $z$ , it is clear that any periodic domain must have terms of the form  $k(D_i - D'_i)$ , where  $k$  is an integer. This implies the weak admissibility of the Heegaard diagram  $(\Sigma, \beta, \alpha, z)$ .

*Remark 4.9.1.* The point  $x_i$  or  $x'_i$  (either one) is a component of the contact class  $c(\xi_{(S,h)}) \in \widehat{HF}(\Sigma, \beta, \alpha, z)$ , where  $\xi_{(S,h)}$  is the contact structure which corresponds to  $(S, h)$ .

**4.9.2. Holomorphic curves in the region  $\mathbb{R} \times [0, 1] \times S_{1/2}$ .** Let  $J \in \mathcal{J}_\Sigma$  with the additional property:

- (&)  $J$  is a product complex structure on  $\mathbb{R} \times [0, 1] \times S_{1/2}$ .

All the holomorphic curves and moduli spaces in this subsection are for the Heegaard diagram  $(\Sigma, \beta, \alpha, z)$ .

**Claim 4.9.2.** *Let  $u \in \widehat{\mathcal{M}}_J(\mathbf{y}, \mathbf{y}')$  for some  $\mathbf{y}, \mathbf{y}' \in \mathcal{S}_{\beta, \alpha}$ . Then the following hold:*

- (1) *If  $u$  is not asymptotic to any  $x''_i$ , then its image is contained in  $\mathbb{R} \times [0, 1] \times S_0$ .*
- (2) *If  $u$  is asymptotic to some  $x''_i$ , then  $u$  has  $x''_i$  at the positive end and a component of  $u$  is either (i) a trivial strip over  $x''_i$  or (ii) a “thin strip” from  $x''_i$  to  $x_i$  or  $x'_i$ , whose projection to  $\Sigma$  is  $D_i$  or  $D'_i$ .*

- (3) If  $u$  is asymptotic to  $x_i$  or  $x'_i$  at the positive end, then a component of  $u$  is a trivial strip over  $x_i$  or  $x'_i$ .

The only nontrivial components of  $u$  which intersect  $\mathbb{R} \times [0, 1] \times S_{1/2}$  are the “thin strips” in (2) and are easily seen to satisfy automatic transversality. Hence a generic  $J$  which satisfies (&) is in  $\mathcal{J}_\Sigma^{reg}$ .

4.9.3. *The variant  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ .* Let  $J \in \mathcal{J}_\Sigma^{reg}$  which satisfies (&). We now define the chain complex  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}), J)$ , which can be defined on a page of the open book  $(S, h)$  and whose homology is isomorphic to  $\widehat{HF}(\Sigma, \beta, \alpha, z, J)$ . The almost complex structure  $J$  will usually be suppressed from the notation.

Let  $\mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$  be the set of  $2g$ -tuples of intersection points of  $\mathbf{a}$  and  $h(\mathbf{a})$ ; equivalently,  $\mathcal{S}_{\mathbf{a}, h(\mathbf{a})} = \{\mathbf{y} \in \mathcal{S}_{\beta, \alpha} \mid \mathbf{y} \subset S_0\}$ . Then we define  $(\widehat{CF}'(S, \mathbf{a}, h(\mathbf{a})), \partial')$  as the subcomplex of  $(\widehat{CF}(\Sigma, \beta, \alpha, z), \partial)$  generated by  $\mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$ . The differential  $\partial$  restricts to  $\partial'$  by Claim 4.9.2(1).

Next define an equivalence relation  $\sim$  on  $\widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))$  as follows: if we write  $\mathbf{y} = \mathbf{y}_0 \cup \mathbf{y}_1$ , where  $\mathbf{y}_0$  consists of chords of type  $x_i, x'_i$ ,  $i = 1, \dots, 2g$ , and  $\mathbf{y}_1$  does not contain any  $x_i, x'_i$ ,  $i = 1, \dots, 2g$ , then  $\mathbf{y} \sim \mathbf{y}'$  if and only if  $\mathbf{y} = \mathbf{y}_0 \cup \mathbf{y}_1$ ,  $\mathbf{y}' = \mathbf{y}'_0 \cup \mathbf{y}'_1$  and  $\mathbf{y}_1 = \mathbf{y}'_1$ . We then take the quotient complex

$$\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) = \widehat{CF}'(S, \mathbf{a}, h(\mathbf{a})) / \sim,$$

with the differential  $\widehat{\partial}$  induced from  $\partial'$ . The differential  $\partial'$  descends to the quotient  $\widehat{\partial}$  by Claim 4.9.2(3).

*Remark 4.9.3.* Since  $\Sigma$  and  $S = S_0$  have opposite orientations, the order  $(\beta, \alpha)$  is switched to  $(\mathbf{a}, h(\mathbf{a}))$ .

The following theorem allows us to restrict from the Heegaard surface  $\Sigma$  to the page  $S$ :

**Theorem 4.9.4.**  $H_*(\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), \widehat{\partial}) \simeq \widehat{HF}(\Sigma, \beta, \alpha, z)$ .

*Proof.* Let us write  $\widehat{CF}$  for  $\widehat{CF}(\Sigma, \beta, \alpha, z)$ . Also let  $\widehat{CF}_k$  be the subgroup of  $\widehat{CF}$  generated by  $2g$ -tuples of chords, exactly  $k$  of which are of the form  $x'_i$ . Using Claim 4.9.2, we can write the differential  $\partial$  on  $\widehat{CF}$  as  $\partial = \partial_0 + \partial_1$ , where  $\partial_0 : \widehat{CF}_k \rightarrow \widehat{CF}_k$  counts  $I_{HF} = 1$  curves whose nontrivial part is contained in  $S_0$  and  $\partial_1 : \widehat{CF}_k \rightarrow \widehat{CF}_{k-1}$  counts  $I_{HF} = 1$  curves whose nontrivial part is contained in  $S_{1/2}$ . In particular,  $\partial_1$  counts HF curves which correspond to the domains  $D_i$  and  $D'_i$ . Since  $\partial^2 = 0$ , it follows that

$$\partial_0^2 = \partial_1^2 = \partial_0 \partial_1 + \partial_1 \partial_0 = 0,$$

i.e.,  $\widehat{CF}$  becomes a double complex.

The  $\partial_1$ -homology of  $\widehat{CF}$  is:

$$H_k(\widehat{CF}, \partial_1) = \begin{cases} \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), & \text{if } k = 0; \\ 0, & \text{if } k > 0. \end{cases}$$

This claim will be proved in Lemma 4.9.5. For the moment we assume it to finish the proof of the theorem. The double complex gives rise to a spectral sequence converging to  $\widehat{HF}(\Sigma, \beta, \alpha, z)$  such that:

$$\begin{aligned} E^1 &= H_*(\widehat{CF}, \partial_1) = \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \\ E^2 &= H_*(H_*(\widehat{CF}, \partial_1), [\partial_0]) = H_*(\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), \widehat{\partial}). \end{aligned}$$

Since  $E^2$  is concentrated in degree  $k = 0$ , the spectral sequence degenerates at the second step and  $\widehat{HF}(\Sigma, \beta, \alpha, z) \cong E^2$ . This proves the theorem.  $\square$

Before proceeding to Lemma 4.9.5, let us introduce some notation. Let  $\mathcal{I}$  be a  $k$ -element subset of  $\{1, \dots, 2g\}$  and let  $\mathcal{I}^c$  be its complement. Then let  $\mathcal{S}_{\mathcal{I}^c}$  be the set of  $(2g - k)$ -tuples  $\mathbf{y}_1$  of chords from  $h(\mathbf{a})$  to  $\mathbf{a}$  such that each  $a_j$  and  $h(a_j)$ ,  $j \in \mathcal{I}^c$ , is used exactly once and no  $x_j, x'_j, x''_j$  is in  $\mathbf{y}_1$ . In particular,  $a_i$  and  $h(a_i)$  remain unoccupied for all  $i \in \mathcal{I}$ .

**Lemma 4.9.5.** *The homology of  $(\widehat{CF}, \partial_1)$  is:*

$$H_k(\widehat{CF}, \partial_1) = \begin{cases} \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), & \text{if } k = 0; \\ 0, & \text{if } k > 0. \end{cases}$$

*Proof.* Let  $(\widehat{CF}(\mathbf{y}_1), \partial_1) \subset (\widehat{CF}, \partial_1)$  be the subcomplex generated by  $2g$ -tuples of chords of the form  $\mathbf{y}_0 \cup \mathbf{y}_1$ , where  $\mathbf{y}_0$  is a  $k$ -tuple of chords consisting of one of  $x_j, x'_j, x''_j$  for each  $j \in \mathcal{I}$  and  $\mathbf{y}_1 \in \mathcal{S}_{\mathcal{I}^c}$ . Since  $(\widehat{CF}, \partial_1)$  is the direct sum of chain complexes of the form  $(\widehat{CF}(\mathbf{y}_1), \partial_1)$ , it suffices to treat each  $(\widehat{CF}(\mathbf{y}_1), \partial_1)$  separately.

Consider the chain complexes  $(C(j), d) = (C_0(j) \oplus C_1(j), d)$ , where

$$C_1(j) = \mathbb{F}\{x''_j\}, \quad C_0(j) = \mathbb{F}\{x_j, x'_j\}, \quad d(x''_j) = x_j - x'_j.$$

The homology groups of those complexes are:

$$(4.9.1) \quad H_k(C(j), d) = \begin{cases} \langle x_j, x'_j \rangle / \langle x_j - x'_j \rangle, & \text{if } k = 0; \\ 0, & \text{if } k = 1. \end{cases}$$

By Claim 4.9.2(2), we have

$$(\widehat{CF}(\mathbf{y}_1), \partial_1) \cong \bigotimes_{j \in \mathcal{I}} (C(j), d).$$

By the Künneth formula,  $H_*(\widehat{CF}(\mathbf{y}_1), \partial_1)$  is generated by the equivalence class  $\{\mathbf{y}'_0 \cup \mathbf{y}_1\}$ , where  $\mathbf{y}_1$  is fixed and  $\mathbf{y}'_0$  ranges over all  $k$ -tuples of chords which consist of one of  $x_j, x'_j$  for each  $j \in \mathcal{I}$ . The lemma then follows.  $\square$

**4.10. Spin<sup>c</sup>-structures.** Let  $\mathcal{S}_{\alpha, \beta}$  be the set of  $k$ -tuples of intersection points of the pointed Heegaard diagram  $(\Sigma, \alpha, \beta, z)$  and let  $\text{Spin}^c(M)$  be the set of Spin<sup>c</sup>-structures on  $M$ . In [OSz1, Section 2.6], Ozsváth and Szabó defined a map

$$s_z : \mathcal{S}_{\alpha, \beta} \rightarrow \text{Spin}^c(M).$$

Although the precise definition of  $s_z$  will not be given here, we review an important property of  $s_z$  which is more or less equivalent to the definition. Given

$\mathbf{y} = \{y_i\}_{i=1}^k, \mathbf{y}' = \{y'_i\}_{i=1}^k \in \mathcal{S}_{\alpha, \beta}$ , the difference between the  $\text{Spin}^c$ -structures corresponding to  $\mathbf{y}$  and  $\mathbf{y}'$  is given by:

$$\epsilon(\mathbf{y}, \mathbf{y}') = PD(s_z(\mathbf{y}) - s_z(\mathbf{y}')) \in H_1(M),$$

where a cycle representing  $\epsilon(\mathbf{y}, \mathbf{y}')$  can be constructed on the Heegaard diagram as follows: For each  $i = 1, \dots, k$ , choose an arc  $\alpha_i^*$  on  $\alpha_i$  from  $y_i$  to  $y'_i$ , where  $y_i, y'_i \in \alpha_i$ . Similarly, we choose arcs  $\beta_i^*$  on  $\beta_i$ ,  $i = 1, \dots, k$ , which connect  $\mathbf{y}'$  to  $\mathbf{y}$ . Then  $\epsilon(\mathbf{y}, \mathbf{y}')$  is the homology class of  $\cup_{i=1}^k (\alpha_i^* \cup \beta_i^*)$ , which is a union of closed curves; see [OSz1, Definition 2.11 and Lemma 2.19]. It is easy to verify that  $\epsilon(\mathbf{y}, \mathbf{y}')$  does not depend on the choice of arcs  $\alpha_i^*$  and  $\beta_i^*$  and provides a topological obstruction to the existence of  $HF$  curves connecting  $\mathbf{y}$  and  $\mathbf{y}'$ .

The Heegaard Floer chain complex  $\widehat{CF}(\Sigma, \alpha, \beta, z)$  therefore splits into a direct sum

$$\widehat{CF}(\Sigma, \alpha, \beta, z) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(M)} \widehat{CF}(\Sigma, \alpha, \beta, z, \mathfrak{s}),$$

where the subgroup  $\widehat{CF}(\Sigma, \alpha, \beta, z, \mathfrak{s})$  is generated by  $\mathbf{y} \in \mathcal{S}_{\alpha, \beta}$  with  $s_z(\mathbf{y}) = \mathfrak{s}$  and is a subcomplex.

We now interpret the above discussion in a way which relates more easily to the splitting of ECH in terms of homology classes of orbit sets. Consider the chain complex  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$  which is generated by the set  $\mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$  of  $2g$ -tuples of intersection points of  $\mathbf{a}$  and  $h(\mathbf{a})$ , i.e., we are restricting to a page  $S$ . The homology groups  $H_1(M; \mathbb{Z}) \simeq H_1(N, \partial N; \mathbb{Z})$  are identified via the isomorphism  $\varpi$ , given in Lemma 2.3.1.

We then define the map

$$\mathfrak{h} : \mathcal{S}_{\mathbf{a}, h(\mathbf{a})} \rightarrow H_1(M)$$

by assigning a cycle  $\mathfrak{h}(\mathbf{y})$  to  $\mathbf{y} = \{y_i\}_{i=1}^{2g} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$  as follows: Suppose  $y_i \in a_i \cap h(a_{\sigma(i)})$  for some  $\sigma \in \mathfrak{S}_{2g}$ . On  $[0, 1] \times S$ , we consider the union of the following oriented arcs:

- $[0, 1] \times \{y_i\}$ ,  $i = 1, \dots, 2g$ , where the orientation is given by  $\partial_t$ ;
- $\{0\} \times c_i$ ,  $i = 1, \dots, 2g$ , where  $c_i$  is a subarc of  $h(a_i)$  which goes from  $h(y_{\sigma(i)})$  to  $y_i$ .

With the identification  $(x, 1) \sim (h(x), 0)$ , the arcs glue to give a cycle in  $N$  which represents  $\mathfrak{h}(\mathbf{y})$ .

**Proposition 4.10.1.** *Let  $\xi$  be the contact structure supported by the open book decomposition  $(S, h)$  of  $M$ , and let  $\mathfrak{s}_\xi$  be the canonical  $\text{Spin}^c$ -structure determined by  $\xi$ . Then for any  $\mathbf{y} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$  we have*

$$s_z(\mathbf{y}) = \mathfrak{s}_\xi + PD(\mathfrak{h}(\mathbf{y})).$$

*Proof.* The equality holds for any  $2g$ -tuple  $\mathbf{x}_0$  which represents the contact class. In fact,  $s_z(\mathbf{x}_0) = \mathfrak{s}_\xi$  by the definition of the contact class and  $\mathfrak{h}(\mathbf{x}_0) = 0$  since the cycle representing it is parallel to  $\partial N$ . Hence, in order to prove the proposition, it suffices to prove that

$$\mathfrak{h}(\mathbf{y}) - \mathfrak{h}(\mathbf{y}') = \epsilon(\mathbf{y}, \mathbf{y}')$$



for all  $\mathbf{y}, \mathbf{y}' \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$ . One can check that  $\mathfrak{h}(\mathbf{y}) - \mathfrak{h}(\mathbf{y}')$  is homologous to the union  $\delta$  of the following types of arcs:

- $[0, 1] \times \{y_i\}$  with orientation  $\partial_t$ ;
- $[0, 1] \times \{y'_i\}$  with orientation  $-\partial_t$ ;
- subarcs of  $\{1\} \times a_i$  connecting from  $\mathbf{y}$  to  $\mathbf{y}'$ ; and
- subarcs of  $\{0\} \times h(a_i)$  connecting from  $\mathbf{y}'$  to  $\mathbf{y}$ .

By homotoping  $\delta$  to a page  $S$ , we see that  $[\delta] = \epsilon(\mathbf{y}, \mathbf{y}')$  with respect to the Heegaard diagram  $(\Sigma, \beta, \alpha, z)$  given in Section 4.9.1.  $\square$

**4.11. Twisted coefficients in Heegaard Floer homology.** In this subsection we review the definition of Heegaard Floer homology with twisted coefficients, originally defined in [OSz2, Section 8], and prove a twisted coefficient analog of Theorem 4.9.4. We describe the construction for  $\widehat{HF}$ ; the construction for  $HF^+$  — which will be used in [CGH3] — can be treated in a similar way.

Fix a  $\text{Spin}^c$ -structure  $\mathfrak{s}$  and a  $k$ -tuple of intersection points  $\mathbf{y}_0$  such that  $s_z(\mathbf{y}_0) = \mathfrak{s}$ . A *complete set of paths for  $\mathfrak{s}$  based at  $\mathbf{y}_0$*  is the choice, for each  $k$ -tuple of intersection points  $\mathbf{y}$  such that  $s_z(\mathbf{y}) = \mathfrak{s}$ , of a surface  $C_{\mathbf{y}}$  which is the projection to  $[0, 1] \times \Sigma$  of a surface representing an element of  $\pi_2(\mathbf{y}, \mathbf{y}_0)$ .<sup>5</sup>

A complete set of paths determines maps

$$\mathfrak{A} : \pi_2(\mathbf{y}, \mathbf{y}') \rightarrow H_2([0, 1] \times \Sigma, \{0\} \times \beta \cup \{1\} \times \alpha) \simeq H_2(M)$$

for all  $\mathbf{y}$  and  $\mathbf{y}'$  such that  $s_z(\mathbf{y}) = s_z(\mathbf{y}') = \mathfrak{s}$  by  $\mathfrak{A}(A) = [C_{\mathbf{y}'} \cup A \cup -C_{\mathbf{y}}]$ . This map is compatible with the action of  $H_2(M)$  on  $\pi_2(\mathbf{y}, \mathbf{y}')$  and with the concatenation of chains with matching ends.

We define

$$\widehat{CF}(\Sigma, \alpha, \beta, z, \mathfrak{s}) = \widehat{CF}(\Sigma, \alpha, \beta, z, \mathfrak{s}) \otimes_{\mathbb{F}} \mathbb{F}[H_2(M; \mathbb{Z})]$$

as an  $\mathbb{F}[H_2(M; \mathbb{Z})]$ -module, with differential

$$\hat{\partial}\mathbf{y} = \sum_{s_z(\mathbf{y}') = \mathfrak{s}} \sum_{A \in \pi_2(\mathbf{y}, \mathbf{y}')} \# \left( \widehat{\mathcal{M}}^{I=1}(\mathbf{y}, \mathbf{y}', A) / \mathbb{R} \right) e^{\mathfrak{A}(A)} \mathbf{y}'.$$

The homology of this complex is the Heegaard Floer homology with twisted coefficients  $\widehat{HF}(M, \mathfrak{s})$ .

Consider the special Heegaard diagram constructed in Section 4.9.1. For every  $\text{Spin}^c$ -structure  $\mathfrak{s} \in \text{Spin}^c(M)$  we define the complex

$$\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}) = \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}) \otimes_{\mathbb{F}} \mathbb{F}[H_2(M; \mathbb{Z})]$$

with the differential induced by the differential on  $\widehat{CF}(\Sigma, \beta, \alpha, z, \mathfrak{s})$ .

**Theorem 4.11.1.**  $H_*(\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}))$  is isomorphic to  $\widehat{HF}(\Sigma, \beta, \alpha, z, \mathfrak{s})$  as  $\mathbb{F}[H_2(M; \mathbb{Z})]$ -modules.

<sup>5</sup>In this section we will identify relative homology classes in  $\tilde{W}$  with relative homology classes in  $[0, 1] \times \Sigma$  when needed.

*Proof.* Fix a distinguished  $2g$ -tuple of generators  $\mathbf{y}_0$  such that  $s_z(\mathbf{y}_0) = \mathfrak{s}$ . We choose a complete set of paths  $C_{\mathbf{y}}$  with the following property: if  $\mathbf{y} = \tilde{\mathbf{y}} \cup \{x_i\}$ ,  $\mathbf{y}' = \tilde{\mathbf{y}} \cup \{x'_i\}$  and  $\mathbf{y}'' = \tilde{\mathbf{y}} \cup \{x''_i\}$ , then  $C_{\mathbf{y}} = C_{\mathbf{y}''} \cup D_i$  and  $C_{\mathbf{y}'} = C_{\mathbf{y}''} \cup D'_i$ , where  $D_i$  and  $D'_i$  are the surfaces corresponding to the thin strips connecting  $x''_i$  to  $x_i$  and  $x'_i$  respectively (see Figure 1). With this choice of a complete set of paths, we have  $\mathfrak{A}(D_i) = \mathfrak{A}(D'_i) = 0$  for all  $i$ , so the proof of Theorem 4.9.4 goes through unchanged.  $\square$

## 5. MODULI SPACES OF MULTISECTIONS

The goal of this section is to introduce the moduli spaces which will be used to define the chain maps

$$\Phi : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \rightarrow PFC_{2g}(N, \alpha_0, \omega)$$

and

$$\Psi : PFC_{2g}(N, \alpha_0, \omega) \rightarrow \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})).$$

The definition of these chain maps can be viewed as a melding of ideas of Seidel [Se1, Se2] and Donaldson-Smith [DS].

Let  $\mathbf{y} = \{y_1, \dots, y_{2g}\}$  be a generator of  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ , where  $y_i \in a_i \cap h(a_{\sigma(i)})$ . Intuitively,  $\mathbf{y}$  is mapped to an element of  $PFC_{2g}(N)$  through an intermediary called a *broken closed string*  $\gamma_{\mathbf{y}}$ . It is a union of closed curves in  $N = S \times [0, 1] / \sim$ , obtained by taking the union of  $y_i \times [0, 1]$ ,  $i = 1, \dots, 2g$ , and  $c_i \times \{0\}$ ,  $i = 1, \dots, 2g$ , where  $c_i$  is a subarc of  $h(a_i)$  which connects  $h(y_{\sigma(i)})$  to  $y_i$ . Note that there is a unique homotopy class of arcs from  $h(y_{\sigma(i)})$  to  $y_i$ , since  $h(a_i)$  is an arc (and not a closed curve). The arcs  $y_i \times [0, 1]$ ,  $c_i \times \{0\}$ ,  $i = 1, \dots, 2g$ , glue up to give a union of closed curves since  $(h(y_{\sigma(i)}), 0) \sim (y_{\sigma(i)}, 1)$ .

**5.1. Symplectic cobordisms.** Recall the stable Hamiltonian structure  $(\alpha_0, \omega)$  on  $N$  from Section 3, where  $\alpha_0$  is given by Equation (3.1.1). For simplicity we assume that  $\alpha_0 = dt$ . The fibration  $N$  is given by:

$$N = (S \times [0, 2]) / \sim, \quad (x, 2) \sim (h(x), 0),$$

where  $h$  is the first return map of the stable Hamiltonian vector field  $R_0 = \partial_t$  with zero flux. Here we make one slight modification: the interval  $[0, 1]$  in Section 2 is now replaced by  $[0, 2]$ . We may assume that  $R_0$  is *Morse-Bott nondegenerate* — i.e., nondegenerate in the interior of  $N$  and Morse-Bott along  $\partial N$  — after a  $C^k$ -small perturbation for  $k \gg 0$ .

*Remark 5.1.1.* Indeed, the stable Hamiltonian vector field  $R_0$  on  $N$  has the same first return map as a Reeb vector field  $R_\tau$ ,  $\tau > 0$ , by construction, and we could have taken  $R_\tau$  to be Morse-Bott nondegenerate.

In this section we introduce the symplectic cobordisms  $(W_+, \Omega_+)$  and  $(W_-, \Omega_-)$ , as well as their “compactifications”  $(\overline{W}_+, \overline{\Omega}_+)$  and  $(\overline{W}_-, \overline{\Omega}_-)$ . The cobordism  $(W_+, \Omega_+)$  interpolates from the stable Hamiltonian structure  $([0, 1] \times S, (dt, \omega))$  at the positive end to the stable Hamiltonian structure  $(N, (\alpha_0, \omega))$  at the negative

end, whereas the cobordism  $(W_-, \Omega_-)$  goes from  $(N, (\alpha_0, \omega))$  at the positive end to  $([0, 1] \times S, (dt, \omega))$  at the negative end.

**5.1.1. The symplectic cobordisms  $(W_+, \Omega_+)$  and  $(W_-, \Omega_-)$ .** Consider the infinite cylinder  $\mathbb{R} \times S^1 \simeq \mathbb{R} \times (\mathbb{R}/2\mathbb{Z})$  with coordinates  $(s, t)$ . Let  $\pi_{S^1} : N \rightarrow S^1$  be the fibration  $(x, t) \mapsto t$  and let  $\pi_{B'} : \mathbb{R} \times N \rightarrow B' = \mathbb{R} \times S^1$  be the extension  $(s, x, t) \mapsto (s, \pi_{S^1}(x, t))$ . Let us write  $N_s = \pi_{B'}^{-1}(\{s\} \times S^1)$  for  $s \in \mathbb{R}$ .

We define  $W_+ = \pi_{B'}^{-1}(B_+)$ , where  $B_+ = (\mathbb{R} \times (\mathbb{R}/2\mathbb{Z})) - B_+^c$  and  $B_+^c$  is the subset  $[2, \infty) \times [1, 2] \subset \mathbb{R} \times (\mathbb{R}/2\mathbb{Z})$  with the corners rounded. See the left-hand side of Figure 2. We write  $\pi_{B_+} : W_+ \rightarrow B_+$  for the restriction of  $\pi_{B'}$ . Note that the boundary of  $W_+$  can be decomposed into two parts that meet along a codimension two corner: the *vertical boundary*  $\partial_v W_+ = \pi_{B_+}^{-1}(\partial B_+)$ , and the *horizontal boundary*  $\partial_h W_+$ , which is equal to the union of the boundaries of the fibers.

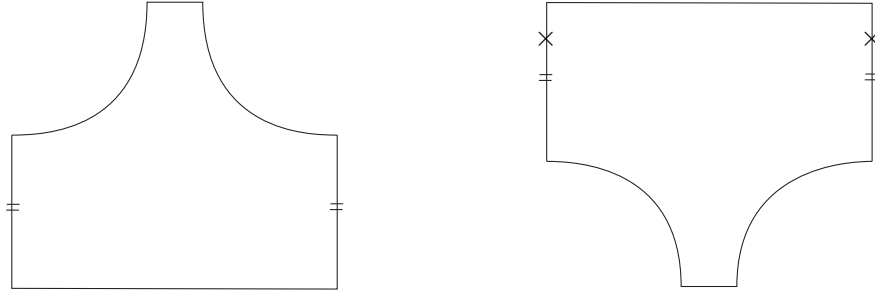


FIGURE 2. The bases  $B_+$  and  $B_-$ . The sides are identified. Both  $B_+$  and  $B_-$  are biholomorphic to a disk with an interior puncture and a boundary puncture. Here the location of  $\overline{m}^b$  on  $B_-$  is indicated by  $\times$ .

Similarly, we define  $W_- = \pi_{B'}^{-1}(B_-)$ , where  $B_- = (\mathbb{R} \times (\mathbb{R}/2\mathbb{Z})) - B_-^c$  and  $B_-^c$  is  $(-\infty, -2] \times [1, 2]$  with the corners rounded. The projection  $\pi_{B_-} : W_- \rightarrow B_-$ , the vertical boundary  $\partial_v W_-$ , and the horizontal boundary  $\partial_h W_-$  are defined analogously.

The symplectic form  $\Omega_+$  (resp.  $\Omega_-$ ) is the restriction of

$$ds \wedge dt + \omega = ds \wedge dt + d_S \beta_t$$

to  $W_+$  (resp.  $W_-$ ). By this we mean the following: On  $\mathbb{R} \times S \times [0, 2]$ , we take the symplectic form  $ds \wedge dt + \omega$ . Then the symplectic form glues under the identification  $(s, x, 2) \sim (s, h(x), 0)$ .

We also write  $cl(B_+)$ ,  $cl(B_-)$  for the compactifications of  $B_+$ ,  $B_-$ , obtained by adjoining the points at infinity  $p_+$  corresponding to  $s = +\infty$ , and  $p_-$  corresponding to  $s = -\infty$ . Therefore  $cl(B_+)$  and  $cl(B_-)$  are isomorphic to the closed unit disk with one marked point on the interior and one marked point on the boundary.

5.1.2. *The cobordisms  $(\overline{W}_+, \overline{\Omega}_+)$  and  $(\overline{W}_-, \overline{\Omega}_-)$ .* We now extend  $(W_+, \Omega_+)$  to  $(\overline{W}_+, \overline{\Omega}_+)$ , which corresponds to capping off each fiber  $S$  by a disk; the definition of  $(\overline{W}_-, \overline{\Omega}_-)$  is analogous.

We first define the capped-off surface  $\overline{S}$ : Let  $D^2 = \{\rho \leq 1\}$  be a disk with polar coordinates  $(\rho, \phi)$ . We write  $z_\infty$  for the origin  $\rho = 0$ . Let  $(y, \theta)$  be the coordinates on a neighborhood  $N(\partial S) \simeq [-\varepsilon, \varepsilon] \times \mathbb{R}/\mathbb{Z}$  of  $\partial S = \{0\} \times \mathbb{R}/\mathbb{Z}$  (inside a slight extension of  $S$ ), as before. Then  $\overline{S} = (S \sqcup D^2)/\sim$ , where  $(y, \theta) \in N(\partial S)$  is identified with  $(\frac{1}{y+1}, -2\pi\theta) \in D^2$ .

Let  $m$  be an integer  $> 2g$ , which we take to be arbitrarily large. We define  $\overline{h}_m : \overline{S} \xrightarrow{\sim} \overline{S}$  to be a smooth extension of  $h : S \xrightarrow{\sim} S$ , depending on  $m$ , such that:

$$\overline{h}_m : D^2 \xrightarrow{\sim} D^2,$$

$$(\rho, \phi) \mapsto (\rho, \phi + \nu_m(\rho)),$$

where  $\nu_m : [0, 1] \rightarrow \mathbb{R}$  is a smooth function which satisfies the following:

- $\nu_m(\rho) = \frac{2\pi}{m}$  for  $\rho \leq \frac{1}{2}$ ;
- $\nu_m(\rho)$  is increasing for  $\frac{1}{2} < \rho < \frac{3}{4}$ ;
- $\nu_m(\rho)$  is decreasing and independent of  $m$  for  $\frac{3}{4} < \rho < 1$ ;
- $\nu_m(\frac{3}{4}) \ll \frac{2\pi}{2g}$  and  $\nu_m(1) = 0$ ;
- $\nu_\infty := \lim_{m \rightarrow \infty} \nu_m$  exists.

In particular,  $\nu_\infty(\rho) = 0$  for  $\rho \leq \frac{1}{2}$ . Taking the limit  $m \rightarrow \infty$  becomes important starting from Section 7.8, but until then we just need  $m \gg 0$  and we simply write  $\overline{h} = \overline{h}_m$  and  $\nu = \nu_m$ .

We then define the suspension

$$\overline{N} = \overline{N}_m = (\overline{S} \times [0, 2]) / (x, 2) \sim (\overline{h}(x), 0).$$

Note that, although  $\overline{N}_m$  depends on  $m$  and  $\nu_m$ , all the  $\overline{N}_m$  are diffeomorphic. The closed manifold  $\overline{N}$  is obtained from  $M$  by a 0-surgery along the binding of the open book. Let  $\overline{\omega}$  be an area form on  $\overline{S}$  which extends  $\omega$  and equals  $\rho d\rho \wedge d\phi$  on  $D^2$ . We then extend the stable Hamiltonian structure  $(\alpha_0 = dt, \omega)$  on  $N$  to the stable Hamiltonian structure  $(\overline{\alpha}_0 = dt, \overline{\omega})$  on  $\overline{N}$ . The Hamiltonian vector field  $\overline{R}_0$  of  $\overline{\omega}$  equals  $\partial_t$  on  $\overline{S} \times [0, 2]$ . Moreover, the orbit  $\delta_0 = \{z_\infty\} \times [0, 2] / \sim$  is the only simple closed orbit of  $\overline{R}_0$  on  $\overline{N} - N$  which intersects  $\overline{S}$  at most  $2g$  times; it is called  $\delta_0$  since it lies on the level set  $\rho = 0$ . The 2-plane field of the stable Hamiltonian structure is  $\ker \overline{\alpha}_0 = T\overline{S}$ .

We now define  $\overline{W}_+$  and  $\overline{W}_-$ . First define extensions  $\overline{W} = \mathbb{R} \times [0, 1] \times \overline{S}$  of  $W = \mathbb{R} \times [0, 1] \times S$  and  $\overline{W}' = \mathbb{R} \times \overline{N}$  of  $W' = \mathbb{R} \times N$ . Let  $\pi_{S^1} : \overline{N} \rightarrow S^1$  be the fibration  $(x, t) \mapsto t$  and let

$$\pi_{B'} : \overline{W}' = \mathbb{R} \times \overline{N} \rightarrow B' = \mathbb{R} \times S^1$$

be the extension  $(s, x, t) \mapsto (s, \pi_{S^1}^{-1}(x, t))$ . For  $*$  = + or -, we set  $\overline{W}_* = \pi_{B'}^{-1}(B_*)$  and write  $\pi_{B_*} : \overline{W}_* \rightarrow B_*$  for the restriction of  $\pi_{B'}^{-1}$  to  $\overline{W}_*$ . The symplectic forms  $\overline{\Omega}_+, \overline{\Omega}_-, \overline{\Omega}, \overline{\Omega}'$  for  $\overline{W}_+, \overline{W}_-, \overline{W}, \overline{W}'$  are obtained from  $ds \wedge dt + \overline{\omega}$  by gluing and restricting as necessary.

Let us write

$$V := \overline{N} - \text{int}(N) = (D^2 \times [0, 2]) / (x, 2) \sim (\overline{h}(x), 0).$$

We then identify  $\varphi : V \xrightarrow{\sim} D^2 \times (\mathbb{R}/2\mathbb{Z})$  via  $(\rho e^{i\phi}, t) \mapsto (\rho e^{i(\phi + tv(\rho)/2)}, t)$ . Note that  $\varphi$  relates two coordinate systems on  $V$ : (i) the “Reeb coordinates”  $(x, t)$  on  $D^2 \times [0, 1]$  such that  $\overline{R}_0 = \partial_t$  and  $(x, 2) \sim (\overline{h}(x), 0)$ , and (ii) the “balanced coordinates” on  $D^2 \times (\mathbb{R}/2\mathbb{Z})$  such that  $\overline{R}_0 = \partial_t + \frac{\nu(\rho)}{2} \partial_\phi$  and  $(x, 2) \sim (x, 0)$ .

For  $*$  = + or −, let

$$\overline{\pi}_{D^2} : \overline{W}_* \cap (\mathbb{R} \times V) \rightarrow D^2$$

be the projection of  $\overline{W}_* \cap (\mathbb{R} \times V)$  to  $V$ , followed by the projection of  $V$  to  $D^2$  via the identification  $\varphi$ , i.e., with respect to the balanced coordinates.

**5.1.3. The marked point  $\overline{\mathbf{m}}$ .** We also choose the marked point  $\overline{\mathbf{m}} = (\overline{\mathbf{m}}^b, \overline{\mathbf{m}}^f) \in \overline{W}_-$ , where  $\overline{\mathbf{m}}^b = (\frac{3}{2}, 0) \in B_-$  and  $\overline{\mathbf{m}}^f = z_\infty \in \overline{S}$ . The marked point will play a crucial role in the definition of the chain map  $\Psi$ .

## 5.2. Lagrangian boundary conditions.

**5.2.1. Lagrangian boundary conditions for  $W_\pm$ .** The symplectic fibration

$$\pi_{B_+} : (W_+, \Omega_+) \rightarrow (B_+, ds \wedge dt)$$

admits a *symplectic connection*, defined as the  $\Omega_+$ -orthogonal of the tangent plane to the fibers. The symplectic connection is spanned by  $\partial_s$  and  $\partial_t$  if we consider  $\Omega = ds \wedge dt + \omega$  on  $\mathbb{R} \times S \times [0, 2]$  before the identification  $(s, x, 2) \sim (s, h(x), 0)$ . (Here  $W_+ \subset (\mathbb{R} \times S \times [0, 2]) / \sim$ .)

We first place a copy of the basis  $\mathbf{a}$  on the fiber  $\pi_{B_+}^{-1}(3, 1)$  and take its parallel transport along  $\partial B_+$  using the symplectic connection. The parallel transport sweeps out a Lagrangian submanifold  $L_{\mathbf{a}}^+$  of  $(W_+, \Omega_+)$ . Let  $L_{a_i}^+$  be the connected component of  $L_{\mathbf{a}}^+$  given by parallel transport of  $a_i$ . Since the symplectic connection is spanned by  $\partial_s$  and  $\partial_t$  on  $\mathbb{R} \times S \times [0, 2]$ , over the strip  $\{s \geq 3, t \in [0, 1]\}$  we have:

$$L_{\mathbf{a}}^+ \cap \{s \geq 3, t = 0\} = \{s \geq 3\} \times h(\mathbf{a}) \times \{0\},$$

$$L_{\mathbf{a}}^+ \cap \{s \geq 3, t = 1\} = \{s \geq 3\} \times \mathbf{a} \times \{1\}.$$

Similarly, the Lagrangian submanifold  $L_{\mathbf{a}}^-$  on the vertical boundary of  $(W_-, \Omega_-)$  is obtained by taking the parallel transport of a copy of  $\mathbf{a}$  — placed on the fiber  $\pi_{B_-}^{-1}(-3, 1)$  — by the symplectic connection.

**5.2.2. Extended Lagrangian boundary conditions.** In what follows we assume that the basis arcs  $a_i$ ,  $i = 1, \dots, 2g$ , depend on  $m \gg 0$  and satisfy some additional conditions. Let  $E \subset \partial D^2$  be the set of endpoints of  $\cup_{i=1, \dots, 2g} a_i$  and let  $y_1(m), \dots, y_{4g}(m)$  be the points of  $E$  in counterclockwise order. Then we assume the following:

- $a_i = a_i(m)$  is oriented;
- $0 < \phi(y_1(m)) < \frac{2\pi}{m}$ ;

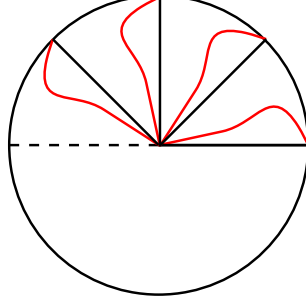


FIGURE 3. Extended arcs in  $D^2$ . The black arcs are portions of the  $\bar{a}_i$  and the red ones are portions of the  $\bar{h}(\bar{a}_i)$ . The dashed arc is one of the  $\vec{a}_{i,j} - \bar{a}_i$ .

- for  $1 \leq j_1, j_2 \leq 4g$ ,

$$\phi^{j_1, j_2}(m) := \phi(y_{j_1}(m)) - \phi(y_{j_2}(m))$$

is an integer multiple of  $\frac{2\pi}{m}$ ;

- there exists  $k_0 \gg 2g$  such that  $\frac{\phi^{j_1, j_2}(m) \cdot m}{2\pi} \geq k_0$  for all  $j_1 > j_2$ ;
- $\lim_{m \rightarrow +\infty} \frac{\phi^{j_1, j_2}(m) \cdot m}{2\pi} = +\infty$  and  $\lim_{m \rightarrow +\infty} \phi^{j_1, j_2}(m) = 0$ ;
- for all quadruples  $j_1 > j_2, j_3 > j_4$ ,  $\lim_{m \rightarrow +\infty} \frac{\phi^{j_1, j_2}(m)}{\phi^{j_3, j_4}(m)} \neq 1$  unless  $(j_1, j_2) = (j_3, j_4)$ .

Assume without loss of generality that the initial point of  $a_i$  is  $x_i$  and the terminal point of  $a_i$  is  $x'_i$ . Let  $\bar{a}_i$  be the (oriented) extension of  $a_i \subset S$  to  $\bar{S}$ , obtained by attaching two radial rays  $\bar{a}_{i,j} = \{0 \leq \rho \leq 1, \phi = \phi_{i,j}\}$ ,  $j = 0, 1$ , where  $\phi_{i,j}$  is a constant. Here  $\bar{a}_{i,0}$  (resp.  $\bar{a}_{i,1}$ ) is the initial (resp. terminal) segment of  $\bar{a}_i$ . We also define the extension  $\vec{a}_{i,j} = \{-1 < \rho \leq 1, \phi = \phi_{i,j}\}$ ,  $j = 0, 1$ , of  $\bar{a}_{i,j} = \{0 \leq \rho \leq 1, \phi = \phi_{i,j}\}$ .

We write  $\bar{\mathbf{a}} = \{\bar{a}_1, \dots, \bar{a}_{2g}\}$ . Then  $L_{\bar{\mathbf{a}}}^\pm$  is the extension of  $L_{\mathbf{a}}^\pm$ , obtained by the parallel transport of a copy of  $\bar{\mathbf{a}}$ , placed at  $\pi_{B_+}^{-1}(3, 1)$  or  $\pi_{B_-}^{-1}(-3, 1)$ . We similarly define  $L_{\hat{\mathbf{a}}}^\pm, L_{\hat{a}_i}^\pm, L_{\vec{a}_i}^\pm, L_{\vec{a}_{i,j}}^\pm$ , and  $L_{\vec{a}_i \cup \vec{a}_{i,j}}^\pm$ , where  $\hat{\mathbf{a}} = \bar{\mathbf{a}} - \{z_\infty\}$  and  $\hat{a}_i = \bar{a}_i - \{z_\infty\}$ .

**Definition 5.2.1.** A bigon (with acute angles) contained in  $D^2$  and bounded by  $\bar{a}_{i,j}$  and  $\bar{h}(\bar{a}_{i,j})$  will be called a *thin strip*. The portion of a thin strip contained in  $D_{1/2} = \{\rho \leq \frac{1}{2}\}$  will be called a *thin wedge*. See Figure 3.

### 5.3. Almost complex structures and moduli spaces for $W, \bar{W}, W'$ , and $\bar{W}'$ .

In this subsection we specify the almost complex structures and moduli spaces for  $W, \bar{W}, W' = \mathbb{R} \times N$ , and  $\bar{W}' = \mathbb{R} \times \bar{N}$ . The notation may conflict with the older ones, namely those used in Sections 3 and 4; in the case of conflict, the new notation supersedes the older ones.

**Convention 5.3.1.** When we write  $\mathbf{y}$  or  $\gamma$  (with possible superscripts, subscripts and other decorations), it is assumed that  $\mathbf{y} \subset S$  and  $\gamma \subset N$ . In particular,  $\mathbf{y}$  and  $\gamma$  do not contain any multiples of  $z_\infty$  or  $\delta_0$ .

**5.3.1. Holomorphic maps to  $W$  and  $\overline{W}$ .** Let  $W = \mathbb{R} \times [0, 1] \times S$  and  $\overline{W} = \mathbb{R} \times [0, 1] \times \overline{S}$ . Also let  $\Omega = ds \wedge dt + \omega$  and  $\overline{\Omega} = ds \wedge dt + \overline{\omega}$ . Then  $\mathcal{J}$  is defined as the set of  $C^\infty$ -smooth  $\Omega$ -admissible almost complex structures on  $W$ .

The analogous space  $\overline{\mathcal{J}}$  for  $\overline{W}$  is slightly more complicated:

**Definition 5.3.2.** Fix  $\varepsilon, \delta > 0$  sufficiently small and  $k > 0$  sufficiently large. Then  $\overline{\mathcal{J}}$  is the set of  $C^\infty$ -smooth  $\overline{\Omega}$ -admissible almost complex structures  $\overline{J}$  on  $\overline{W}$  which satisfy the following:

- (1) there exists an  $\overline{\Omega}$ -admissible almost complex structure  $\overline{J}_0$  which restricts to the standard complex structure on the subsurface  $D^2 \subset \overline{S}$  of each fiber;
- (2)  $\overline{J} = \overline{J}_0$  on  $\{\rho \leq \varepsilon\} \cup W$  and  $|\overline{J} - \overline{J}_0|_k < \delta$  over  $\overline{W}$ .

Here  $|\cdot|_k$  is some fixed  $C^k$ -norm.

We denote by  $J$  the restriction of  $\overline{J}$  to  $W$ . Let  $\mathcal{S} = \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$  be the set of  $2g$ -tuples of intersection points on  $\mathbf{a} \cap h(\mathbf{a})$ . Given  $\mathbf{y}, \mathbf{y}' \in \mathcal{S}$  and  $J \in \mathcal{J}$ , let  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  be the moduli space of HF curves from  $\mathbf{y}$  to  $\mathbf{y}'$  with respect to  $J$  which are contained in  $\mathbb{R} \times [0, 1] \times S$ .

We next discuss holomorphic curves in  $\overline{W}$ .

**Definition 5.3.3.** Let  $\mathbf{y}, \mathbf{y}'$  be  $k$ -tuples of  $\mathbf{a} \cap h(\mathbf{a})$  and  $\overline{J} \in \overline{\mathcal{J}}$ . Then a *degree  $k \leq 2g$  multisection  $\overline{u}$  from  $\mathbf{y}$  to  $\mathbf{y}'$  in  $(\overline{W}, \overline{J})$*  is a holomorphic map  $\overline{u} : \dot{F} \rightarrow \overline{W}$  which is a degree  $k$  multisection of  $\pi_B : \overline{W} \rightarrow B$ , satisfies the conditions of Definition 4.3.1 with  $L_\alpha$  and  $L_\beta$  replaced by  $L_{\overline{\mathbf{a}}} = \mathbb{R} \times \{1\} \times \overline{\mathbf{a}}$  and  $L_{\overline{h}(\overline{\mathbf{a}})} = \mathbb{R} \times \{0\} \times \overline{h}(\overline{\mathbf{a}})$ , and is asymptotic to  $\mathbf{y}$  and  $\mathbf{y}'$  at the positive and negative ends.

**Definition 5.3.4.** The *section at infinity*  $\sigma_\infty$  is the map  $\mathbb{R} \times [0, 1] \rightarrow \overline{W}$  which sends  $(s, t) \mapsto (s, t, z_\infty)$ .

The section at infinity is  $\overline{J}$ -holomorphic for every  $\overline{J} \in \overline{\mathcal{J}}$ . We will be sloppy with the notation and routinely identify  $\sigma_\infty$  with its image.

Let  $z_\infty^\dagger \in D_{1/2} \subset \overline{S}$  be a point with  $\rho$ -coordinate less than  $\frac{1}{2}$  and in the complement of all arcs  $\overline{a}_{i,j}$  and  $\overline{h}(\overline{a}_{i,j})$ . The orbit  $\mathcal{O}(z_\infty^\dagger)$  of  $z_\infty^\dagger$  under the action of  $\overline{h}$  consists of  $m$  points, and each thin wedge in  $D_{1/2}$  between  $\overline{a}_{i,j}$  and  $\overline{h}(\overline{a}_{i,j})$  contains exactly one point of  $\mathcal{O}(z_\infty^\dagger)$ . Let  $\sigma_\infty^\dagger = \mathbb{R} \times [0, 1] \times \mathcal{O}(z_\infty^\dagger)$ . (Note that this multisection does not have Lagrangian boundary conditions; it will only be used to impose topological constraints on the HF curves.)

**Definition 5.3.5.** Given a degree  $k$  multisection  $\overline{u}$  of  $\overline{W}$ , we define  $n(\overline{u}) = \langle \overline{u}, \sigma_\infty^\dagger \rangle$ , where  $\langle \cdot, \cdot \rangle$  denotes the algebraic intersection between the images.

The intersection number  $n(\overline{u})$  is a homological invariant since  $z_\infty^\dagger$  was chosen so that  $\mathcal{O}(z_\infty^\dagger)$  is disjoint from the Lagrangian arcs.

**Lemma 5.3.6.** *The intersection number  $n(\overline{u})$  satisfies the following properties:*

- (1)  $n(\bar{u}) \geq 0$  and  $n(\bar{u}) = 0$  if and only if the image of  $\bar{u}$  is disjoint from  $\sigma_\infty^\dagger$ ;
- (2) if  $\langle \bar{u}, \sigma_\infty \rangle > 0$ , then  $n(\bar{u}) \geq m$ ;
- (3) if  $n(\bar{u}) = 1$ , then  $\bar{u}$  projects onto a thin strip; and
- (4)  $n(\bar{u})$  is independent of the choice of  $z_\infty^\dagger$ .

*Proof.* (1) follows from the positivity of intersections of pseudo-holomorphic maps in dimension four, (2) is a consequence of the fact that holomorphic maps are open, and (3) is a consequence of the positivity of intersections and of the fact that every thin wedge contains only one point of  $\mathcal{O}(z_\infty^\dagger)$ . Finally, (4) follows from the fact that different choices for  $\sigma_\infty^\dagger$  are connected by a path of multisections which are disjoint from the Lagrangian boundary conditions.  $\square$

**Definition 5.3.7.** Let  $\mathbf{y}$  and  $\mathbf{y}'$  be  $k$ -tuples of  $\mathbf{a} \cap h(\mathbf{a})$ . Then  $\mathcal{M}_{\bar{J}}(\mathbf{y}, \mathbf{y}')$  is the moduli space of degree  $k$  multisections  $\bar{u}$  of  $(\bar{W}, \bar{J})$  from  $\mathbf{y}$  to  $\mathbf{y}'$ .

*Modifiers.* For any moduli space  $\mathcal{M}_{\star_1}(\star_2)$  we may place modifiers  $*$  as in  $\mathcal{M}_{\star_1}^*(\star_2)$  to denote the subset of  $\mathcal{M}_{\star_1}(\star_2)$  satisfying  $*$ . Typical self-explanatory modifiers are  $I = i$ ,  $n = m$ , and  $\deg = k$ . (Note however that the degree can be inferred from  $\star_2$ .)

The following lemma is an easy consequence of Lemma 5.3.6, in particular of points (1) and (4).

**Lemma 5.3.8.**  $\mathcal{M}_{\bar{J}}^{n=0}(\mathbf{y}, \mathbf{y}') = \mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  if  $\mathbf{y}, \mathbf{y}' \in \mathcal{S}$ .

**5.3.2. Holomorphic maps to  $W'$  and  $\bar{W}'$ .** Let  $\mathcal{J}'$  be the space of  $C^\infty$ -smooth  $(\alpha_0, \omega)$ -adapted almost complex structures on  $W' = \mathbb{R} \times N$ . The definition of the analogous space  $\bar{\mathcal{J}}'$  for  $\bar{W}' = \mathbb{R} \times \bar{N}$  is slightly more complicated, but completely analogous to Definition 5.3.2:

**Definition 5.3.9.** Fix  $\varepsilon, \delta > 0$  sufficiently small and  $k > 0$  sufficiently large. Then  $\bar{\mathcal{J}}'$  is the set of  $C^\infty$ -smooth  $(\bar{\alpha}_0, \bar{\omega})$ -adapted almost complex structures  $\bar{J}'$  on  $\mathbb{R} \times \bar{N}$  which satisfy the following:

- (1) there exists an  $(\bar{\alpha}_0, \bar{\omega})$ -adapted almost complex structure  $\bar{J}'_0$  which restricts to the standard complex structure on the subsurface  $D^2 \subset \bar{S}$  of each fiber;
- (2)  $\bar{J}' = \bar{J}'_0$  on  $\{\rho \leq \varepsilon\} \cup (\mathbb{R} \times N)$  and  $|\bar{J}' - \bar{J}'_0|_k < \delta$  over  $\mathbb{R} \times \bar{N}$ .

Here  $|\cdot|_k$  is some fixed  $C^k$ -norm.

Let  $\hat{\mathcal{P}}$  be the set of orbits of  $R_0$  in  $\text{int}(N)$ , together with  $e$  and  $h$  on  $\partial N$ , and let  $\bar{\mathcal{P}} = \hat{\mathcal{P}} \cup \{\delta_0\}$ . Let  $\hat{\mathcal{O}}_k$  and  $\bar{\mathcal{O}}_k$  be the set of orbit sets constructed respectively from  $\hat{\mathcal{P}}$  or  $\bar{\mathcal{P}}$  which intersect  $\bar{S} \times \{0\}$  exactly  $k$  times. A *PFH curve* is a  $\bar{J}'$ -holomorphic Morse-Bott building in  $\mathbb{R} \times \bar{N}$  without fiber components, which connect two orbit sets in  $\bar{\mathcal{O}}_{2g}$ . If we perturb the Morse-Bott family, then the perturbed PFH curve (if it exists) can be viewed as a degree  $2g$   $\bar{J}'$ -holomorphic multisection of the holomorphic fibration  $\mathbb{R} \times \bar{N} \rightarrow \mathbb{R} \times S^1$ .

We denote the *section at infinity*  $\mathbb{R} \times \delta_0$  by  $\sigma'_\infty$ . Choose a point  $z_\infty^\dagger \in \bar{S}$  which is sufficiently close to  $z_\infty$ . Then there is a periodic orbit  $\delta_0^\dagger$  of  $\bar{R}_0$  with period  $m$  which passes through  $z_\infty^\dagger$ . We then write  $(\sigma'_\infty)^\dagger = \mathbb{R} \times \delta_0^\dagger$ .



**Definition 5.3.10.** Given a PFH curve  $\bar{u}$  in  $\overline{W'}$ , we define  $n'(\bar{u}) = \langle \bar{u}, (\sigma'_\infty)^\dagger \rangle$ .

The proof of the following is similar to that of Lemma 5.3.6.

**Lemma 5.3.11.** *The intersection number  $n'(\bar{u})$  satisfies the following properties:*

- (1)  $n'(\bar{u}) \geq 0$  and  $n'(\bar{u}) = 0$  if and only if the image of  $\bar{u}$  is disjoint from  $(\sigma'_\infty)^\dagger$ ;
- (2) if  $\langle \bar{u}, \mathbb{R} \times \delta_0 \rangle > 0$ , then  $n'(\bar{u}) > 0$ ; and
- (3)  $n'(\bar{u})$  is independent of the choice of  $z_\infty^\dagger$ .

**Definition 5.3.12.** Given  $\gamma, \gamma' \in \widehat{\mathcal{O}}_k$  and  $J' \in \mathcal{J}'$ , let  $\mathcal{M}_{J'}(\gamma, \gamma')$  be the moduli space of  $(W', J')$ -holomorphic Morse-Bott buildings from  $\gamma$  to  $\gamma'$  without fiber components. Given  $\delta_0^p \gamma, \delta_0^q \gamma' \in \overline{\mathcal{O}}_k$  and  $\overline{J'} \in \overline{\mathcal{J}'}$ , let  $\mathcal{M}_{\overline{J'}}(\delta_0^p \gamma, \delta_0^q \gamma')$  be the moduli space of  $(\overline{W'}, \overline{J'})$ -holomorphic Morse-Bott buildings from  $\gamma$  to  $\gamma'$  without fiber components.

We also use the modifier  $* = s$  to indicate the subset of somewhere injective curves.

**Lemma 5.3.13.** *If  $\gamma$  and  $\gamma'$  are contained in  $N$ , then  $\mathcal{M}_{\overline{J'}}^{n'=0}(\gamma, \gamma') = \mathcal{M}_{J'}(\gamma, \gamma')$ .*

*Proof.* Let  $\bar{u}$  be a  $\overline{J'}$ -holomorphic Morse-Bott building satisfying the constraint  $n'(\bar{u}) = 0$ . Let  $T_{\rho_0}$  be the torus  $\{\rho = \rho_0\} \subset \overline{N}$ , oriented as the boundary of the solid torus  $\{\rho \leq \rho_0\}$ . The torus  $T_{\rho_0}$  is foliated by orbits of the Hamiltonian vector field  $\overline{R}_0$  and, for a dense set of  $\rho_0$ , those orbits are closed. If  $\delta_{\rho_0}$  is one such orbit, then the homology class  $[\delta_{\rho_0}]$  is a rational multiple of  $[\delta_0^\dagger]$  in  $H_2(\overline{N} - \gamma \cup \gamma')$ . This implies that  $\langle \bar{u}, \mathbb{R} \times \delta_{\rho_0} \rangle = 0$  for  $\rho_0$  in a dense set. The positivity of intersections then implies that the image of  $\bar{u}$  is contained in  $N$ .  $\square$

#### 5.4. Almost complex structures and moduli spaces for $W_+$ , $\overline{W}_+$ , $W_-$ , and $\overline{W}_-$ .

##### 5.4.1. Almost complex structures.

**Definition 5.4.1.** An almost complex structure  $\overline{J}_+$  on  $\overline{W}_+$  is *admissible* if it is the restriction to  $\overline{W}_+$  of an almost complex structure  $\overline{J'} \in \overline{\mathcal{J}'}$  on  $\overline{W'}$ . If  $\overline{J}_+$  agrees with  $\overline{J}$  (resp.  $\overline{J'}$ ) at the positive (resp. negative) end, then  $\overline{J}_+$  is *compatible* with  $\overline{J}$  (resp.  $\overline{J'}$ ).

An *admissible* almost complex structure  $J_+$  on  $W_+$  is the restriction of an admissible almost complex structure on  $\overline{W}_+$ . The admissibility criteria for  $J_-$  and  $\overline{J}_-$  on  $W_-$  and  $\overline{W}_-$  are analogous.

The space of  $C^\infty$ -smooth admissible almost complex structures  $J_\pm$  (resp.  $\overline{J}_\pm$ ) on  $W_\pm$  (resp.  $\overline{W}_\pm$ ) will be denoted by  $\mathcal{J}_\pm$  (resp.  $\overline{\mathcal{J}}_\pm$ ).

The restrictions of  $\sigma'_\infty = \mathbb{R} \times \delta_0 \subset \overline{W'}$  to  $\overline{W}_+$  and  $\overline{W}_-$ , respectively, are holomorphic sections at infinity, written as  $\sigma_\infty^+$  and  $\sigma_\infty^-$ .

**Remark 5.4.2.** If  $J_+ \in \mathcal{J}_+$ , then the projection  $\pi_{B_+} : W_+ \rightarrow B_+$  is holomorphic. This is due to the fact that the fibers  $\{(s, t)\} \times S$  are holomorphic and  $J_+$  takes  $\partial_s$  to  $\partial_t$ . The same holds for  $\overline{J}_+$ ,  $J_-$ , and  $\overline{J}_-$ .

**5.4.2. Moduli spaces for  $W_+$ .** Let  $(F, j)$  be a compact Riemann surface, possibly disconnected, with an  $l$ -tuple of punctures  $\mathbf{p} = \{p_1, \dots, p_l\}$  in the interior of  $F$  and a  $k$ -tuple of punctures  $\mathbf{q} = \{q_1, \dots, q_k\}$  on  $\partial F$ , such that (i) each component of  $F$  has nonempty boundary and at least one interior puncture and (ii) each component of  $\partial F$  has at least one boundary puncture. We write  $\dot{F} = F - \mathbf{p} - \mathbf{q}$  and  $\partial \dot{F} = \partial F - \mathbf{q}$ .

Let  $\mathbf{y}$  be a  $k$ -tuple of  $\mathbf{a} \cap h(\mathbf{a})$  and let  $\gamma = \prod_j \gamma_j^{m_j}$  be an orbit set in  $\hat{\mathcal{O}}_k$ .

**Definition 5.4.3.** Let  $J_+ \in \mathcal{J}_+$ . Then a *degree  $k \leq 2g$  multisection of  $(W_+, J_+)$  from  $\mathbf{y}$  to  $\gamma$*  is either (i) a holomorphic map

$$u : (\dot{F}, j) \rightarrow (W_+, J_+)$$

which is a degree  $k$  multisection of  $\pi_{B_+} : W_+ \rightarrow B_+$  and which additionally satisfies the following:

- (1)  $u(\partial \dot{F}) \subset L_{\mathbf{a}}^+$  and  $u$  maps each connected component of  $\partial \dot{F}$  to a different  $L_{a_i}^+$ ;
- (2)  $\lim_{w \rightarrow q_i} \pi_{\mathbb{R}} \circ u(w) = +\infty$  and  $\lim_{w \rightarrow p_i} \pi_{\mathbb{R}} \circ u(w) = -\infty$ ;
- (3)  $u$  converges to a strip over  $[0, 1] \times \mathbf{y}$  near  $\mathbf{q}$ ;
- (4)  $u$  converges to a cylinder over a multiple of some  $\gamma_j$  near each puncture  $p_i$  so that the total multiplicity of  $\gamma_j$  over all the  $p_i$ 's is  $m_j$ ;
- (5) the energy of  $u$  given by Equation (4.3.1) is finite;

or is (ii) a Morse-Bott building which, after perturbing  $R_0$  using an appropriate Morse function and perturbing  $J_+$ , becomes a degree  $k$  multisection of  $\pi_{B_+} : W_+ \rightarrow B_+$  which additionally satisfies (1)–(5). Here  $\pi_{\mathbb{R}}$  is the projection  $\pi_{B_+} : W_+ \rightarrow B_+ \subset \mathbb{R} \times S^1$ , followed by the projection to  $\mathbb{R}$ .

A  $(W_+, J_+)$ -curve is a degree  $2g$  multisection of  $(W_+, J_+)$ .

The finiteness of the Hofer energy  $E(u)$  implies that  $u$  is a cylinder over a Reeb chord or a closed orbit in neighborhoods of punctures  $p_i$  and  $q_i$ . Hence (5) implies (3) and (4) for some  $\mathbf{y}$  and  $\gamma$ . Moreover, since the orbits are nondegenerate, the convergence is exponential by the work of Abbas [Ab] for chords and Hofer-Wysocki-Zehnder [HWZ1] for closed orbits.

*Remark 5.4.4.* For all practical purposes, it suffices to assume that the Morse-Bott family on  $\partial N$  has been perturbed into a pair  $h, e$  of nondegenerate orbits and that  $J_+ \in \mathcal{J}_+$  with respect to the perturbed Reeb vector field.

Let  $\check{W}_+$  be  $W_+$  with the ends  $\{s > 3\}$  and  $\{s < -1\}$  removed. We can view  $\check{W}_+$  as a compactification of  $W_+$ , obtained by attaching  $[0, 1] \times S$  at  $s = +\infty$  and  $N$  at  $s = -\infty$ . We define  $Z_{\mathbf{y}, \gamma} \subset \check{W}_+$  as the subset

$$Z_{\mathbf{y}, \gamma} = (L_{\mathbf{a}}^+ \cap \check{W}_+) \cup (\{3\} \times [0, 1] \times \mathbf{y}) \cup (\{-1\} \times \gamma).$$

As in Section 4.3, the  $W_+$ -curve  $u : \dot{F} \rightarrow W_+$  can be compactified to a continuous map

$$\check{u} : (\check{F}, \partial \check{F}) \rightarrow (\check{W}_+, Z_{\mathbf{y}, \gamma}).$$

We write  $\mathcal{M}_{J_+}(\mathbf{y}, \gamma)$  for the moduli space of multisections of  $(W_+, J_+)$  from  $\mathbf{y}$  to  $\gamma$ . We denote by  $H_2(W_+, \mathbf{y}, \gamma)$  the equivalence classes of continuous degree  $2g$  multisections in  $W_+$  satisfying Conditions (1)–(4) of Definition 5.4.3, where two multisections are equivalent if they represent the same element in  $H_2(\check{W}_+, Z_{\mathbf{y}, \gamma})$ . Then

$$\mathcal{M}_{J_+}(\mathbf{y}, \gamma) = \coprod_{A \in H_2(\check{W}_+, \mathbf{y}, \gamma)} \mathcal{M}_{J_+}(\mathbf{y}, \gamma, A).$$

**5.4.3. Moduli spaces for  $\overline{W}_+$ .** Let  $\mathbf{y}$  be a  $k$ -tuple of  $\mathbf{a} \cap h(\mathbf{a})$  and let  $\gamma \in \widehat{\mathcal{O}}_k$ . Then a degree  $k \leq 2g$  multisection of  $(\overline{W}_+, \overline{J}_+)$  from  $\mathbf{y}$  to  $\gamma$  is defined as in Definition 5.4.3, where  $W_+$ ,  $J_+$ ,  $L_{a_i}^+$  are replaced by  $\overline{W}_+$ ,  $\overline{J}_+$ ,  $L_{\widehat{a}_i}^+$ .<sup>6</sup> We write  $\mathcal{M}_{\overline{J}_+}(\mathbf{y}, \gamma)$  for the moduli space of multisections of  $(\overline{W}_+, \overline{J}_+)$  from  $\mathbf{y}$  to  $\gamma$ .

Let  $\delta_0^\dagger$  be the closed orbit of  $\overline{R}_0$  used in Definition 5.3.10. We define  $(\sigma_\infty^+)^\dagger$  as the restriction of  $\mathbb{R} \times \delta_0^\dagger$  to  $\overline{W}_+$ .

**Definition 5.4.5.** Given a multisection  $\overline{u}$  of  $\overline{W}_+$ , we define  $n^+(\overline{u}) = \langle \overline{u}, (\sigma_\infty^+)^\dagger \rangle$ .

**Lemma 5.4.6.** *The intersection number  $n^+(\overline{u})$  satisfies the following properties:*

- (1)  $n^+(\overline{u}) \geq 0$  and  $n^+(\overline{u}) = 0$  if and only if the image of  $\overline{u}$  is disjoint from  $(\sigma_\infty^+)^\dagger$ ;
- (2) if  $\langle \overline{u}, \sigma_\infty^+ \rangle > 0$ , then  $n^+(\overline{u}) \geq m$ ; moreover  $n^+(\overline{u}) = m$  if and only if there is a unique transverse intersection point between the image of  $\overline{u}$  and  $\sigma_\infty^+$ ; and
- (3)  $n^+(\overline{u})$  is independent of the choice of  $\delta_0^\dagger$ .

*Proof.* The proof is similar to that of Lemma 5.3.6. The only difference is in (2), which we discuss in more detail. Consider  $\overline{u} : \dot{F} \rightarrow \overline{W}_+$  and let  $p \in \dot{F}$  be a point such that  $\overline{u}(p) \in \sigma_\infty^+$ . If  $\pi_{D^2}$  is the projection of a neighborhood of  $\overline{u}(p) \in \overline{W}_+$  to  $D^2 \subset \overline{S}$  along the symplectic connection, then  $\pi_{D^2} \circ \overline{u}$  is holomorphic and nonconstant, and therefore maps an open neighborhood of  $p$  in  $\dot{F}$  to an open neighborhood of  $z_\infty$ . This implies that  $n^+(\overline{u}) \geq d \cdot m$ , where  $d$  is the multiplicity of the intersection between the image of  $\overline{u}$  and  $\sigma_\infty^+$ .  $\square$

**Lemma 5.4.7.** *If  $\overline{u} \in \mathcal{M}_{\overline{J}_+}^{n^+=0}(\mathbf{y}, \gamma)$ , then  $\text{Im}(\overline{u}) \subset W_+$ .*

The proof is similar to that of Lemma 5.3.13 and will be omitted.

**5.4.4. Moduli spaces for  $\overline{W}_-$ .** The moduli space of holomorphic maps which is used to define the map  $\Psi$  from  $\widehat{PFC}$  to  $\widehat{CF}$  is of a slightly different type from the moduli space which is used to define the map  $\Phi$  from  $\widehat{CF}$  to  $\widehat{PFC}$ . In particular, the target of the holomorphic maps is  $\overline{W}_-$  instead of  $W_-$ . The reason we need to consider more complicated holomorphic curves instead of curves analogous to  $W_+$ -curves is that the naive  $W_-$ -curves do not have the desired Fredholm index. See the index calculations in Section 5.5.2 and Remark 5.5.6 for an explanation.

<sup>6</sup>We require that components of  $\partial \dot{F}$  be mapped to  $L_{\widehat{a}_i}^+$  and be disjoint from  $L_{\widehat{a}_i}^+ - L_{\widehat{a}_i}^+$ .

Let  $(F, j)$  be a compact Riemann surface, possibly disconnected, with an  $l$ -tuple of punctures  $\mathbf{p} = \{p_1, \dots, p_l\}$  in the interior of  $F$  and a  $k$ -tuple of punctures  $\mathbf{q} = \{q_1, \dots, q_k\}$  on  $\partial F$ , such that (i) each component of  $F$  has nonempty boundary and has at least one interior puncture and (ii) each component of  $\partial F$  has at least one boundary puncture. We write  $\dot{F} = F - \mathbf{p} - \mathbf{q}$  and  $\partial \dot{F} = \partial F - \mathbf{q}$ .

Let  $\mathbf{y}$  be a  $k$ -tuple of  $\mathbf{a} \cap h(\mathbf{a})$  and let  $\gamma = \prod_j \gamma_j^{m_j} \in \hat{O}_k$ .

**Definition 5.4.8.** Let  $\bar{J}_- \in \bar{\mathcal{J}}_-$ . Then a *degree  $k \leq 2g$  multisection of  $(\bar{W}_-, \bar{J}_-)$  from  $\mathbf{y}$  to  $\gamma$*  is either (i) a holomorphic map

$$\bar{u} : (\dot{F}, j) \rightarrow (\bar{W}_-, \bar{J}_-)$$

which is a degree  $k$  multisection of  $\pi_{B_-} : \bar{W}_- \rightarrow B_-$  and which additionally satisfies the following:

- (1)  $\bar{u}(\partial \dot{F}) \subset L_{\mathbf{a}}^-$  and  $\bar{u}$  maps each connected component of  $\partial \dot{F}$  to a different  $L_{\hat{a}_i}^-$ ;
- (2)  $\lim_{w \rightarrow q_i} \pi_{\mathbb{R}} \circ \bar{u}(w) = -\infty$  and  $\lim_{w \rightarrow p_i} \pi_{\mathbb{R}} \circ \bar{u}(w) = +\infty$ ;
- (3)  $\bar{u}$  converges to strip over  $[0, 1] \times \mathbf{y}$  near  $\mathbf{q}$ ;
- (4)  $\bar{u}$  converges to a cylinder over a multiple of some  $\gamma_j$  near each puncture  $p_i$  so that the total multiplicity of  $\gamma_j$  over all the  $p_i$ 's is  $m_j$ ;
- (5) the energy of  $\bar{u}$  given by Equation (4.3.1) is finite;

or is (ii) a Morse-Bott building which, after perturbing  $\bar{R}_0$  using an appropriate Morse function and perturbing  $\bar{J}_-$ , becomes a degree  $k$  multisection of  $\pi_{B_-} : \bar{W}_- \rightarrow B_-$  which additionally satisfies (1)–(5). Here  $\pi_{\mathbb{R}}$  is the projection  $\pi_{B_-} : \bar{W}_- \rightarrow B_- \subset \mathbb{R} \times S^1$ , followed by the projection to  $\mathbb{R}$ .

Let  $\delta_0^\dagger$  be the closed orbit of  $\bar{R}_0$  used in Definition 5.3.10. We define  $(\sigma_\infty^-)^\dagger$  as the restriction of  $\mathbb{R} \times \delta_0^\dagger$  to  $\bar{W}_-$ .

**Definition 5.4.9.** Given a multisection  $\bar{u}$  of  $\bar{W}_-$ , we define  $n^-(\bar{u}) = \langle \bar{u}, (\sigma_\infty^-)^\dagger \rangle$ .

The proof of the following is similar to that of Lemma 5.4.6.

**Lemma 5.4.10.** *The intersection number  $n^-(\bar{u})$  satisfies the following properties:*

- (1)  $n^-(\bar{u}) \geq 0$  and  $n^-(\bar{u}) = 0$  if and only if the image of  $\bar{u}$  is disjoint from  $(\sigma_\infty^-)^\dagger$ ;
- (2) if  $\langle \bar{u}, \sigma_\infty^- \rangle > 0$ , then  $n^-(\bar{u}) \geq m$ ; moreover  $n^-(\bar{u}) = m$  if and only if there is a unique transverse intersection point between the image of  $\bar{u}$  and  $\sigma_\infty^-$ ; and
- (3)  $n^-(\bar{u})$  is independent of the choice of  $\delta_0^\dagger$ .

**Definition 5.4.11.** A  $(\bar{W}_-, \bar{J}_-)$ -curve is a degree  $2g$  multisection of  $(\bar{W}_-, \bar{J}_-)$  satisfying  $n^-(\bar{u}) = m$ .

Let  $\check{\bar{W}}_-$  be  $\bar{W}_-$  with the ends  $\{s > 1\}$  and  $\{s < -3\}$  removed. We can view  $\check{\bar{W}}_-$  as a compactification of  $\bar{W}_-$ , obtained by attaching  $[0, 1] \times \bar{S}$  at  $s = -\infty$  and  $\bar{N}$  at  $s = +\infty$ . Also let

$$Z_{\gamma, \mathbf{y}} = (L_{\mathbf{a}}^- \cap \check{\bar{W}}_-) \cup (\{1\} \times \gamma) \cup (\{-3\} \times [0, 1] \times \mathbf{y}).$$

We write  $\mathcal{M}_{\overline{J}_-}(\gamma, \mathbf{y})$  for the moduli space of multisections of  $(\overline{W}_-, \overline{J}_-)$  from  $\gamma$  to  $\mathbf{y}$ . We denote by  $H_2(\overline{W}_-, \gamma, \mathbf{y})$  the equivalence classes of continuous degree  $2g$  multisections in  $\overline{W}_-$  satisfying Conditions (1)–(4) of Definition 5.4.8, where two multisections are equivalent if they represent the same element in  $H_2(\overline{W}_-, Z_{\gamma, \mathbf{y}})$ . Then

$$\mathcal{M}_{\overline{J}_-}(\gamma, \mathbf{y}) = \coprod_{A \in H_2(\overline{W}_-, \gamma, \mathbf{y})} \mathcal{M}_{\overline{J}_-}(\gamma, \mathbf{y}, A).$$

Also let  $\mathcal{M}_{\overline{J}_-}(\gamma, \mathbf{y}; \overline{\mathbf{m}}) \subset \mathcal{M}_{\overline{J}_-}(\gamma, \mathbf{y})$  be the subset of curves which pass through the marked point  $\overline{\mathbf{m}}$ .

The following lemma is similar to Lemma 5.4.7:

**Lemma 5.4.12.** *If  $\overline{u} \in \mathcal{M}_{\overline{J}_-}^{n=0}(\gamma, \mathbf{y})$ , then  $\text{Im}(\overline{u}) \subset W_-$ .*

**5.5. The Fredholm index.** In this subsection we compute the Fredholm index  $\text{ind}_{W_+}$  of a  $(W_+, J_+)$ -curve from  $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$  to  $\gamma = \prod \gamma_j^{m_j} \in \widehat{\mathcal{O}}_{2g}$  and the Fredholm index  $\text{ind}_{\overline{W}_-}$  of a  $(\overline{W}_-, \overline{J}_-)$ -curve from  $\gamma$  to  $\mathbf{y}$ .

The Fredholm indices are computed using the doubling technique of Hofer-Lizan-Sikorav [HLS], which we quickly review, referring to the original paper for the details.

**5.5.1. Doubling.** Let  $(F, j)$  be a compact Riemann surface with boundary,  $\mathbf{p}$  and  $\mathbf{q}$  be finite sets of interior and boundary punctures of  $F$ , and  $\dot{F} = F - \mathbf{p} - \mathbf{q}$ . We form the *double*  $(2\dot{F}, 2j)$  of  $(\dot{F}, j)$  by gluing two copies of  $\dot{F}$  with opposite complex structures  $j$  and  $-j$  along their boundary  $\partial\dot{F} = \partial F - \mathbf{q}$ . By the Schwarz reflection principle, the two complex structures match and the doubled surface becomes a punctured Riemann surface.

Let  $E \rightarrow \dot{F}$  be a complex vector bundle with fiberwise complex structure  $i$  and let  $L \rightarrow \partial\dot{F}$  be a totally real subbundle of maximal rank. Let  $\overline{E} \rightarrow \dot{F}$  be a complex vector bundle whose fiber  $\overline{E}_p$  at  $p \in \dot{F}$  is  $E_p$  with complex structure  $-i$ . We then construct the doubled complex vector bundle  $2E \rightarrow 2\dot{F}$  by gluing  $E \rightarrow \dot{F}$  and  $\overline{E} \rightarrow \dot{F}$  along  $\partial\dot{F}$  such that at each  $p \in \partial\dot{F}$  the gluing map identifies the fibers  $(E_p, i) \simeq (\overline{E}_p, -i)$  via an involution which fixes  $L_p$  pointwise. Let  $\sigma : 2\dot{F} \xrightarrow{\sim} 2\dot{F}$  be the anti-holomorphic involution  $\sigma$  which switches  $(\dot{F}, j)$  and  $(\dot{F}, -j)$  and let  $\tilde{\sigma} : 2E \xrightarrow{\sim} 2E$  be the anti- $\mathbb{C}$ -linear bundle isomorphism which projects to  $\sigma$  and identifies  $E_p \simeq \overline{E}_{\sigma(p)}$  by the identity map, where  $p \in (\dot{F}, j)$ . Finally, given a linear Cauchy-Riemann type operator  $D$  on  $E$ , we can define the doubled operator  $2D$  on  $2E$  with the property that  $2D$  is  $\tilde{\sigma}$ -invariant and its restriction to  $E \rightarrow \Sigma$  is  $D$ .

One of the results of [HLS] is the following:

**Theorem 5.5.1.** *Suppose that both  $D$  and  $2D$  are Fredholm operators in some suitable Sobolev spaces. Then:*

- $\text{ind}(D) = \frac{1}{2} \text{ind}(2D)$ ; and
- if  $2D$  is surjective, then  $D$  is also surjective.

Our situation is slightly more general than that considered by [HLS], since we are considering boundary punctures and exponential weights. The proof, however, remains largely unmodified.

**5.5.2. The  $W_+$  case.** For the purposes of computing indices, we replace  $W_+$  by  $\check{W}_+$ , as defined in Section 5.4.2. The tangent space  $T\check{W}_+$  splits into the vertical and horizontal subbundles  $TS$  and  $TB_+$  via the symplectic connection. (In this section we slightly abuse notation and write  $TS$  for the vertical subbundle over  $\check{W}_+$ .)

We define a trivialization  $\tau$  of  $TS$  along  $Z_{\mathbf{y},\gamma}$  as follows: First define  $\tau$  along  $L_{\mathbf{a}}^+ \cap \check{W}_+$  by orienting all  $\mathbf{a}$ -arcs arbitrarily as in Section 4.4.2 and extending the trivialization by parallel transport along  $\partial_v W_+$ . We then extend the trivialization of  $TS|_{L_{\mathbf{a}}^+ \cap \check{W}_+}$  in an arbitrary manner to a trivialization of  $TS$  along  $\{3\} \times [0, 1] \times \mathbf{y}$  and along  $\{-1\} \times \gamma$ .

Let  $u : (\check{F}, j) \rightarrow (W_+, J_+)$  be a  $W_+$ -curve. Suppose the negative ends of  $u$  partition  $m_j$  into  $(m_{j1}, m_{j2}, \dots)$ . We then write:

$$\mu_\tau(\gamma, u) = \sum_j \sum_i \mu_\tau(\gamma_j^{m_{ji}}),$$

where  $\mu_\tau(\gamma_j^{m_{ji}})$  is the Conley-Zehnder index of the  $m_{ji}$ -fold cover of  $\gamma_j$  with respect to  $\tau$ .

We now define a real rank one subbundle  $\mathcal{L}_0$  of  $TS$  along

$$(L_{\mathbf{a}}^+ \cap \check{W}_+) \cup (\{3\} \times [0, 1] \times \mathbf{y}).$$

Let  $\mathcal{L}_0 = TL_{\mathbf{a}}^+ \cap TS$  on  $L_{\mathbf{a}}^+ \cap \check{W}_+$ . In particular,  $\mathcal{L}_0 = T\mathbf{a}(y_i)$  at  $\{3\} \times \{1\} \times \{y_i\}$  and  $\mathcal{L}_0 = Th(\mathbf{a})(y_i)$  at  $\{3\} \times \{0\} \times \{y_i\}$ . We then extend  $\mathcal{L}_0$  across  $\{3\} \times [0, 1] \times \mathbf{y}$  by rotating in the counterclockwise direction from  $Th(\mathbf{a})$  to  $T\mathbf{a}$  in  $TS$  by the minimum amount possible. Assuming orthogonal intersections between  $\mathbf{a}$  and  $h(\mathbf{a})$ , the angle of rotation is  $\frac{\pi}{2}$ .

Let  $\mu_\tau(y_i)$  be the Maslov index of  $\mathcal{L}_0$  along  $\{3\} \times [0, 1] \times \{y_i\}$  with respect to  $\tau$ . If  $\mathcal{L} = \check{u}^* \mathcal{L}_0$ , then we define  $\mu_\tau(\mathbf{y})$  to be the Maslov index of  $\mathcal{L}$  with respect to  $\tau$ . By the definitions of  $\mathcal{L}_0$  and  $\tau$ , it is immediate that

$$\mu_\tau(\mathbf{y}) = \sum_{i=1}^{2g} \mu_\tau(y_i).$$

We then have the following Fredholm index formula for  $W_+$ -curves:

**Proposition 5.5.2** (Fredholm index formula for  $W_+$ -curves). *The Fredholm index of a  $(W_+, J_+)$ -curve  $u : (\check{F}, j) \rightarrow (W_+, J_+)$  from  $\mathbf{y}$  to  $\gamma$  is given by the formula:*

$$(5.5.1) \quad \text{ind}_{W_+}(u) = -\chi(\check{F}) - 2g + \mu_\tau(\mathbf{y}) - \mu_\tau(\gamma, u) + 2c_1(u^*TS, \tau).$$

*Proof.* We double the surface  $\check{F}$  along  $\partial\check{F} = \partial F - \mathbf{q}$ , where  $\mathbf{q}$  is the set of boundary punctures, and double the pullback bundle  $u^*TW_+$  along the real subbundle

$(u|_{\partial\dot{F}})^*TL_{\mathbf{a}}^+$  on  $\partial\dot{F}$ , to obtain the doubled surface  $2\dot{F}$  with a complex vector bundle  $2u^*TW_+ \rightarrow 2\dot{F}$ . The boundary punctures  $q_i$  of  $\dot{F}$  are doubled to give positive interior punctures  $2q_i$  on  $2\dot{F}$ .

Since the trivialization  $\tau$  of  $TS|_{Z_{\mathbf{y},\gamma}}$  was chosen to be tangent to  $TL_{\mathbf{a}}^+ \cap TS$ , its pullback to  $u^*TW_+$  is compatible with the doubling operation and gives a trivialization  $2\tau$  of  $2u^*TS$  over a neighborhood of the punctures of  $2\dot{F}$ . Also let  $\tau'$  be a partially defined trivialization of  $TB_+$  which is given by  $\partial_s$  at the positive and negative ends of  $B_+$ . Then  $\tau'$  can be doubled to  $2\tau'$  on  $2u^*TB_+$  which is defined over a neighborhood of the punctures of  $2\dot{F}$ .

The linearized  $\bar{\partial}$ -operator at  $u$  splits as a sum  $D_u = D_u^0 \oplus K_u$ , where  $D_u^0$  is a Cauchy-Riemann type operator on  $W^{1,p}$ -sections of  $u^*TW_+$  with exponential weights and  $K_u$  is a direct sum of finite-dimensional operators, one for each puncture of  $\dot{F}$ , with index 2 for interior punctures and index 1 for boundary punctures; see [Dr, Section 2.3] for the definition of the operator  $K_u$ . We define  $2D_u = 2D_u^0 \oplus K'_u$ , where  $K'_u$  is a finite-dimensional operator and contributes by 2 to the index for each puncture of  $2\dot{F}$ . Technically speaking,  $K'_u$  is not the double of  $K_u$ ; nevertheless  $\text{ind}(K_u) = \frac{1}{2} \text{ind}(K'_u)$ . By applying Theorem 5.5.1 to  $D_u^0$ , we obtain  $\text{ind}(D_u) = \frac{1}{2} \text{ind}(2D_u)$ .

We now apply the standard Fredholm index formula (see for example Dragnev [Dr]) for holomorphic curves in symplectizations — with slight modifications — to obtain:

$$(5.5.2) \quad \begin{aligned} \text{ind}(2D_u) = & -\chi(2\dot{F}) + \mu_{2\tau}(2\mathbf{y}) - 2\mu_{2\tau}(\gamma, 2u) \\ & + 2c_1(2u^*TB_+, 2\tau') + 2c_1(2u^*TS, 2\tau). \end{aligned}$$

Here  $\mu_{2\tau}(2\mathbf{y})$  is the sum of the Conley-Zehnder indices, computed with respect to the trivialization  $2\tau$ , of the paths of symplectic matrices arising from the asymptotic operators at  $2y_i$ . The last term is the relative first Chern class of the double of the pullback bundle  $u^*TS$  with respect to  $2\tau$ . The one term that is not present in the Fredholm index formula for  $J$ -holomorphic curves in a symplectization is the penultimate term  $2c_1(2u^*TB_+, 2\tau')$ .

We then compute the following:

- (a)  $\chi(2\dot{F}) = 2\chi(\dot{F}) - 2g$ ;
- (b)  $\mu_{2\tau}(2y_i) = 2\mu_{\tau}(y_i) - 1$ ;
- (c)  $\mu_{2\tau}(\gamma, 2u) = 2\mu_{\tau}(\gamma, u)$ ;
- (d)  $c_1(2u^*TB_+, 2\tau') = -2g$ ;
- (e)  $c_1(2u^*TS, 2\tau) = 2c_1(u^*TS, \tau)$ .

(a), (c) and (e) are straightforward. (b) will be proved in Lemma 5.5.3 below. (d) is equivalent to  $2g$  times the Euler characteristic of the double of  $B_+$ , by arguing in a manner similar to that of Claim 4.5.10.

Summarizing, we obtain:

$$\begin{aligned} \text{ind}_{W_+}(u) &= (-\chi(\dot{F}) + g) + (\mu_{\tau}(\mathbf{y}) - g) - \mu_{\tau}(\gamma, u) - 2g + 2c_1(u^*TS, \tau) \\ &= -\chi(\dot{F}) - 2g + \mu_{\tau}(\mathbf{y}) - \mu_{\tau}(\gamma, u) + 2c_1(u^*TS, \tau). \end{aligned}$$

This completes the proof of the proposition.  $\square$

**Lemma 5.5.3.**  $\mu_{2\tau}(2y_i) = 2\mu_\tau(y_i) - 1$ .

*Proof.* Let  $q_i \in \partial F$  be a (positive) boundary puncture and let  $\mathcal{E} = [0, \infty) \times [0, 1] \subset \dot{F}$  be a strip-like end with coordinates  $(s, t)$  which parametrizes a neighborhood of  $q_i$ . Suppose  $u$  maps the end asymptotically to  $y_i$ . Fix a symplectic trivialization

$$\Theta : u|_{\mathcal{E}}^* TS \xrightarrow{\sim} \mathbb{R}^2 \times \mathcal{E}$$

such that  $u|_{\partial\mathcal{E}}^* (TL_{\mathbf{a}}^+ \cap TS)$  corresponds to  $(\mathbb{R} \oplus 0) \times \partial\mathcal{E}$ . Here  $\partial\mathcal{E} = \partial\dot{F} \cap \mathcal{E}$ .

Let  $2\mathcal{E} = ([0, \infty) \times [0, 2]) / (s, 0) \sim (s, 2)$  be the end of the doubled surface  $2\dot{F}$ , which corresponds to the interior puncture  $2q_i$  and is obtained by doubling  $\mathcal{E}$ . The involution  $\sigma$  is given by  $\sigma(s, t) = (s, 2 - t)$  with respect to these coordinates. Similarly, we have a symplectic trivialization

$$2\Theta : 2u|_{2\mathcal{E}}^* TS \xrightarrow{\sim} \mathbb{R}^2 \times 2\mathcal{E}$$

and  $\tilde{\sigma}$  is given by  $\tilde{\sigma}((x_1, x_2), s, t) = ((x_1, -x_2), s, 2 - t)$ .

Let  $J_0 \partial_t + S_i(t)$  be the *asymptotic operator* of  $D_u^0$  corresponding to  $q_i$  with respect to the trivialization  $\Theta$ . Here  $J_0$  is the standard complex structure on  $\mathbb{R}^2$  and  $S_i(t)$  is a symmetric  $2 \times 2$  matrix with real coefficients. For the definition of the asymptotic operator and its relation with the Fredholm theory of linearized  $\bar{\partial}$ -operators, see [Dr, Section 3] or [HT2].<sup>7</sup> The solution of the Cauchy problem

$$\dot{\Phi}(t) = J_0 S_i(t) \Phi(t), \quad \Phi(0) = 0$$

is a path of symplectic matrices which represents the linearized Reeb flow along the chord  $y_i$ , expressed with respect to the trivialization  $\tau$ . In our setting, we may assume that  $\Phi(t)$  is a path of unitary matrices. By identifying  $(\mathbb{R}^2, J_0) = (\mathbb{C}, i)$ , we write  $\Phi(t) = e^{i\alpha(t)}$  for some function  $\alpha : [0, 1] \rightarrow \mathbb{R}$ . Then

$$\mu_\tau(y_i) = \left\lfloor \frac{\alpha(1) - \alpha(0)}{\pi} \right\rfloor + 1,$$

where  $\lfloor x \rfloor$  is the greatest integer  $\leq x$ .

The double of the asymptotic operator, i.e., the asymptotic operator of the doubled operator  $2D_u^0$  at the interior puncture  $2q_i$ , can be written as  $J_0 \partial_t + \tilde{S}_i(t)$ , where:

$$\tilde{S}_i(t) = \begin{cases} S_i(t), & t \in [0, 1]; \\ C S_i(2 - t) C^{-1}, & t \in [1, 2], \end{cases}$$

and  $C = \text{diag}(1, -1)$ . The solution of the corresponding Cauchy problem is

$$\tilde{\Phi}(t) = \begin{cases} \Phi(t), & t \in [0, 1]; \\ C \Phi(2 - t) \Phi(1)^{-1} C^{-1} \Phi(1), & t \in [1, 2]. \end{cases}$$

Hence can write  $\tilde{\Phi}(t) = e^{i\tilde{\alpha}(t)}$ , where  $\tilde{\alpha} : [0, 2] \rightarrow \mathbb{R}$  is given by:

$$\tilde{\alpha}(t) = \begin{cases} \alpha(t), & t \in [0, 1]; \\ -\alpha(2 - t) + 2\alpha(1), & t \in [1, 2]. \end{cases}$$

The Conley-Zehnder index of the path  $\tilde{\Phi}$  is

$$\mu_{2\tau}(2y_i) = 2 \left\lfloor \frac{\tilde{\alpha}(2) - \tilde{\alpha}(0)}{2\pi} \right\rfloor + 1.$$

---

<sup>7</sup>What we call  $S_i$  here is written as  $C_{2\infty}^i$  in [Dr].



Since

$$\begin{aligned} \left\lfloor \frac{\tilde{\alpha}(2) - \tilde{\alpha}(0)}{2\pi} \right\rfloor &= \left\lfloor \frac{(\tilde{\alpha}(2) - \tilde{\alpha}(1)) + (\tilde{\alpha}(1) - \tilde{\alpha}(0))}{2\pi} \right\rfloor \\ &= \left\lfloor \frac{2(\tilde{\alpha}(1) - \tilde{\alpha}(0))}{2\pi} \right\rfloor = \left\lfloor \frac{\alpha(1) - \alpha(0)}{\pi} \right\rfloor, \end{aligned}$$

we obtain  $\mu_{2\tau}(2y_i) = 2\mu_\tau(y_i) - 1$ , as desired.  $\square$

**5.5.3. The  $\overline{W}_-$  case.** The trivialization  $\tau$  of  $T\overline{S}_{\overline{W}_-}$  is defined on  $Z_{\gamma, \mathbf{y}}$  in a manner similar to that of  $W_+$ , except that the positive and negative ends are reversed. The definition of the real rank one subbundle  $\mathcal{L}$  of  $T\overline{S}$  along  $(L_{\tilde{\mathbf{a}}}^- \cap \overline{W}_-) \cup (\{-3\} \times [0, 1] \times \mathbf{y})$  is also similar and will be omitted.

If  $\overline{u}$  is a  $\overline{W}_-$ -curve from  $\gamma$  to  $\mathbf{y}$ , then the Fredholm index  $\text{ind}_{\overline{W}_-}(\overline{u})$  is defined as the expected dimension of the moduli space  $\mathcal{M}_{\overline{J}_-}(\gamma, \mathbf{y})$  near  $\overline{u}$ .

*Remark 5.5.4.* The expected dimension of  $\mathcal{M}_{\overline{J}_-}(\gamma, \mathbf{y}; \overline{\mathbf{m}})$  is  $\text{ind}_{\overline{W}_-}(\overline{u}) - 2$ .

We then have the following Fredholm index formula:

**Proposition 5.5.5** (Fredholm index formula for  $\overline{W}_-$ -curves). *The Fredholm index of a  $(\overline{W}_-, \overline{J}_-)$ -curve  $\overline{u} : (\dot{F}, j) \rightarrow (\overline{W}_-, \overline{J}_-)$  from  $\gamma$  to  $\mathbf{y}$  is given by the formula:*

$$(5.5.3) \quad \text{ind}_{\overline{W}_-}(\overline{u}) = -\chi(\dot{F}) + \mu_\tau(\gamma, \overline{u}) - \mu_\tau(\mathbf{y}) + 2c_1(\overline{u}^*T\overline{S}, \tau).$$

*Proof.* By a calculation similar to that of Proposition 5.5.2, we obtain

$$\text{ind}_{\overline{W}_-}(\overline{u}) = (-\chi(\dot{F}) + g) + \mu_\tau(\gamma, \overline{u}) - (\mu_\tau(\mathbf{y}) - g) - 2g + 2c_1(\overline{u}^*T\overline{S}, \tau),$$

which simplifies to the desired result.  $\square$

*Remark 5.5.6* (Reason for considering  $\overline{W}_-$ ). We will give a rough explanation of the reason for considering holomorphic curves in  $\overline{W}_-$  which pass through  $\overline{\mathbf{m}}$ . Suppose we have a  $W_+$ -curve  $u$  from  $\mathbf{y}$  to  $\gamma$  and a  $W_-$ -curve  $v$  from  $\gamma$  to  $\mathbf{y}$  (i.e.,  $\text{Im}(v) \subset W_-$ ). Then, by taking the sum of Equations (5.5.1) and (5.5.3), we compute the Fredholm index of the glued curve  $u \# v$ , corresponding to the stacking of  $W_+$  at the top ( $s > 0$ ) and  $W_-$  at the bottom ( $s < 0$ ), to be:

$$\text{ind}(u \# v) = -\chi(\dot{F}) + 2c_1((u \# v)^*TS, \tau) - 2g,$$

where  $\dot{F}$  is obtained from gluing the domains of  $u$  and  $v$ . The stacking gives rise to a chain map  $\widehat{CF} \rightarrow \widehat{CF}$ , which we expect to map  $\mathbf{y} \mapsto \mathbf{y}$  via restrictions of trivial cylinders (modulo chain homotopy). This would mean  $\chi(\dot{F}) = 0$  and  $c_1((u \# v)^*TS, \tau) = 0$ . This leaves us with a deficiency of  $2g$ . Introducing the point constraint at  $\overline{\mathbf{m}}$ , from the perspective of Fredholm indices, is basically equivalent to applying a multiple connected sum to the holomorphic curve  $v$  and the fiber  $\overline{S}$  which passes through  $\overline{\mathbf{m}}$ . The multiple connected sum is performed at the  $2g$  intersection points between  $v$  and  $\overline{S}$ . We effectively increase the Fredholm index by  $2g + 2$ , obtained by adding up the following contributions:

- (i) the Fredholm index of the fiber  $\overline{S}$ , which is  $\chi(\overline{S}) = 2 - 2g$ ;
- (ii)  $2g$  intersection points, each of which contributes  $+2$  to minus the Euler characteristic.

The point constraint then cuts the expected dimension of the moduli space by  $-2$ , for a net gain of  $2g$ .

**5.6. The ECH index.** In this section we present the ECH index  $I_{W_+}$ , the relative adjunction formula, and the ECH index inequality for  $W_+$ . The situation for  $\overline{W}_-$  is analogous, and will not be discussed explicitly, except to point out some differences.

**5.6.1. Definitions.** Let  $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$  and  $\gamma = \prod_{j=1}^l \gamma_j^{m_j} \in \widehat{\mathcal{O}}_{2g}$ . Let  $\tau$  be a partial trivialization of  $TS_{\check{W}_+}$  along  $Z_{\mathbf{y}, \gamma}$  as given in Section 5.5.2. Using the trivialization  $\tau$ , for each simple orbit  $\gamma_j$  of  $\gamma$ , we choose an identification of a sufficiently small neighborhood  $N(\gamma_j)$  of  $\gamma_j$  with  $\gamma_j \times D^2$ , where  $D^2$  has polar coordinates  $(r, \theta)$ .

**Definition 5.6.1** ( $\tau$ -trivial representative). A  $\tau$ -trivial representative  $\check{C}$  of  $A \in H_2(W_+, \mathbf{y}, \gamma)$  is an oriented immersed compact surface in the class  $A$  which satisfies the following:

- (1)  $\check{C}$  is embedded on  $\check{W}_+ - \{s = -1\}$ ;
- (2)  $\check{C}$  is positively transverse to the fibers  $\{(s, t)\} \times \overline{S}$  along all of  $\partial\check{C}$ ;
- (3)  $\check{C}$  is  $\tau$ -trivial in the sense of Definition 4.5.2 at the HF end;
- (4)  $\check{C}$  is  $\tau$ -trivial at the ECH end, i.e., for all sufficiently small  $\varepsilon > 0$ ,  $\check{C} \cap \{s = -1 + \varepsilon\}$  consists of  $m_j$  disjoint circles  $\{r = \varepsilon, \theta = \theta_{ji}\}$ ,  $i = 1, \dots, m_j$ , in  $N(\gamma_j)$  for all  $j$ . (See [Hu1, Definition 2.3].)

Let  $\partial_+\check{C} = \partial\check{C} \cap \{s > 0\}$  and  $\partial_-\check{C} = \partial\check{C} \cap \{s < 0\}$ .

**Definition 5.6.2** (Relative intersection form). Let  $A \in H_2(W_+, \mathbf{y}, \gamma)$  be a homology class which is realized by a  $\tau$ -trivial representative  $\check{C}$ . Then the relative intersection form  $Q_\tau(A)$  is given by  $\langle \check{C}, \check{C}' \rangle$ , where  $\check{C}'$  is a pushoff of  $\check{C}$  which satisfies the following:

- (1)  $\check{C}'$  is pushed off in the  $J_+\tau$ -direction along  $\partial_+\check{C}$ ; and
- (2) for small  $\varepsilon > 0$ ,  $\check{C}' \cap \{s = -1 + \varepsilon\}$  consists of  $m_j$  disjoint circles  $\{r = \varepsilon, \theta = \theta_{ji} + \varepsilon'\}$ ,  $i = 1, \dots, m_j$ , in  $N(\gamma_j)$  for all  $j$ , where  $\varepsilon' > 0$  is a sufficiently small constant.

(See [Hu1, Definition 2.4].)

**5.6.2. Relative adjunction formula.** Let  $u : \dot{F} \rightarrow W_+$  be a  $W_+$ -curve and let  $\tilde{u} : \dot{F} \rightarrow \check{W}_+$  be its compactification. Then we write  $w_\tau^-(u)$  for the total writhe of the braids  $u(\dot{F}) \cap \{s = s_0\}$ ,  $s_0 \ll 0$ , with respect to  $\tau$ . Similarly, if  $\overline{u} : \dot{F} \rightarrow \overline{W}_-$  is a  $\overline{W}_-$ -curve, then we write  $w_\tau^+(\overline{u})$  for the total writhe of the braids  $\overline{u}(\dot{F}) \cap \{s = s_0\}$ ,  $s_0 \gg 0$ , with respect to  $\tau$ .

**Lemma 5.6.3** (Relative adjunction formula). Let  $u : \dot{F} \rightarrow W_+$  be a  $W_+$ -curve in the homology class  $A \in H_2(W_+, \mathbf{y}, \gamma)$ . Then

$$(5.6.1) \quad c_1(u^*TW_+, (\tau, \partial_t)) = \chi(\dot{F}) - w_\tau^-(u) + Q_\tau(A) - 2\delta(u)$$

where  $\partial_t$  trivializes  $TB_+$ .

For a  $\overline{W}_-$ -curve  $\overline{u}$ , we replace  $-w_\tau^-(u)$  by  $w_\tau^+(\overline{u})$ .

*Proof.* Suppose  $u$  is immersed. If  $\nu$  is the normal bundle of  $u$ , then we have the formula:

$$(5.6.2) \quad c_1(u^*TW_+, (\tau, \partial_t)) = c_1(T\check{F}, \partial_t) + c_1(\nu, \tau).$$

In the general case, we combine the calculations of Lemma 4.5.8 and [Hu1, Proposition 3.1] to obtain the equation

$$c_1(u^*TW_+, (\tau, \partial_t)) = c_1(T\check{F}, \partial_t) - w_\tau^-(u) + Q_\tau(A) - 2\delta(u).$$

The analog of Claim 4.5.10 for the present situation is

$$(5.6.3) \quad c_1(T\check{F}, \partial_t) = \chi(\check{F}).$$

The difference in the shape of the base (i.e.,  $B$  vs.  $B_+$ ) accounts for the discrepancy between Equation (5.6.3) and Claim 4.5.10.  $\square$

*Remark 5.6.4.* Since  $c_1(u^*TB_+, \partial_t) = 0$ , we have

$$c_1(u^*TW_+, (\tau, \partial_t)) = c_1(u^*TS, \tau) + c_1(u^*TB_+, \partial_t) = c_1(u^*TS, \tau).$$

5.6.3. *ECH index.* We now define the ECH indices for  $W_+$  and  $\overline{W}_-$ .

**Definition 5.6.5** (ECH index for  $W_+$ ). Given a class  $A \in H_2(W_+, \mathbf{y}, \gamma)$  which admits a  $\tau$ -trivial representative  $\check{C}$ , we define

$$(5.6.4) \quad I_{W_+}(A) = c_1(T\check{W}_+|_A, (\tau, \partial_t)) + Q_\tau(A) + \mu_\tau(\mathbf{y}) - \tilde{\mu}_\tau(\gamma) - 2g,$$

where  $\tilde{\mu}_\tau(\gamma)$  is the symmetric Conley-Zehnder index at the negative (ECH) end.

**Definition 5.6.6** (ECH index for  $\overline{W}_-$ ). Given a class  $A \in H_2(\overline{W}_-, \gamma, \mathbf{y})$  which admits a  $\tau$ -trivial representative  $\check{C}$ , we define

$$(5.6.5) \quad I_{\overline{W}_-}(A) = c_1(T\check{\overline{W}}_-|_A, (\tau, \partial_t)) + Q_\tau(A) + \tilde{\mu}_\tau(\gamma) - \mu_\tau(\mathbf{y}).$$

As usual, the ECH indices  $I_{W_+}(A)$  and  $I_{\overline{W}_-}(A)$  are independent of the choice of trivialization  $\tau$ .

*Remark 5.6.7.* To obtain a finite count of  $\overline{W}_-$ -curves which pass through the point  $\overline{\mathbf{m}}$ , we count curves  $\overline{u}$  with ECH index  $I_{\overline{W}_-}(\overline{u}) = 2$ .

5.6.4. *Additivity of indices.*

**Lemma 5.6.8** (Additivity of indices). *If  $u \in \mathcal{M}_J(\mathbf{y}, \mathbf{y}')$ ,  $v \in \mathcal{M}_{J_+}(\mathbf{y}', \gamma)$ , and  $u \# v$  is a pre-glued curve, then*

$$(5.6.6) \quad \text{ind}_{W_+}(u \# v) = \text{ind}_{HF}(u) + \text{ind}_{W_+}(v),$$

$$(5.6.7) \quad I_{W_+}(u \# v) = I_{HF}(u) + I_{W_+}(v).$$

*Similarly, if  $u \in \mathcal{M}_{J_+}(\mathbf{y}, \gamma)$ ,  $v \in \mathcal{M}_{J'}(\gamma, \gamma')$ , and  $u \# v$  is a pre-glued curve, then*

$$(5.6.8) \quad \text{ind}_{W_+}(u \# v) = \text{ind}_{W_+}(u) + \text{ind}_{ECH}(v),$$

$$(5.6.9) \quad I_{W_+}(u \# v) = I_{W_+}(u) + I_{ECH}(v).$$

*Proof.* The additivity for  $\text{ind}$  is well-known and the additivity for  $I$  is immediate from the definitions.  $\square$

**5.6.5. Index inequality.** Although we will not define it here, given an integer  $m_k > 0$  and a simple orbit  $\gamma_k$ , we can define the *incoming partition*  $P_{\gamma_k}^{in}(m_k)$  and the *outgoing partition*  $P_{\gamma_k}^{out}(m_k)$  as in [Hu2, Definition 4.14].

We have the following index inequality, which is analogous to [Hu1, Theorem 1.7] (also see [Hu2, Theorem 4.15] which is applicable to symplectic cobordisms). Note that a  $W_+$ -curve is automatically simply-covered.

**Theorem 5.6.9** (Index inequality). *Let  $u$  be a  $W_+$ -curve from  $\mathbf{y}$  to  $\gamma = \prod_{k=1}^l \gamma_k^{m_k}$ . If the negative ends of  $u$  partition  $m_k$  into  $(m_{k1}, m_{k2}, \dots)$ , then*

$$\text{ind}_{W_+}(u) + 2\delta(u) \leq I_{W_+}(u),$$

*and equality holds only if  $P_{\gamma_k}^{in}(m_k) = (m_{k1}, m_{k2}, \dots)$  for all  $k$ . Similarly, if  $\bar{u}$  is a  $\bar{W}_-$ -curve from  $\gamma = \prod_{k=1}^l \gamma_k^{m_k}$  to  $\mathbf{y}$  and the positive ends of  $u$  partition  $m_k$  into  $(m_{k1}, m_{k2}, \dots)$ , then*

$$\text{ind}_{W_-}(\bar{u}) + 2\delta(\bar{u}) \leq I_{W_-}(\bar{u}),$$

*and equality holds only if  $P_{\gamma_k}^{out}(m_k) = (m_{k1}, m_{k2}, \dots)$  for all  $k$ .*

Here  $\delta(u)$  is the signed count of interior singularities of  $u$ , where each singularity contributes positively to  $\delta(u)$ . The following are immediate: (i) if  $\text{ind}_{W_+}(u) = I_{W_+}(u)$ , then  $u$  is embedded; and (ii) if  $I_{W_+}(u) = 0$  or  $1$ , then  $u$  is embedded.

*Proof.* If we plug Equation (5.5.1), Equation (5.6.4), and the relative adjunction formula (5.6.1) into  $I_{W_+}(u) - \text{ind}_{W_+}(u)$ , we obtain:

$$I_{W_+}(u) - \text{ind}_{W_+}(u) = \mu_\tau(\gamma, u) + w_\tau^-(u) - \tilde{\mu}_\tau(\gamma) + 2\delta(u).$$

The statement for  $W_+$  then follows from the writhe inequality

$$w_\tau^-(u) \geq \tilde{\mu}_\tau(\gamma) - \mu_\tau(\gamma, u),$$

where equality holds if and only if the partition  $(m_{k1}, m_{k2}, \dots)$  of the negative end of  $u$  coincides with the incoming partition  $P_{\gamma_k}^{in}(m_k)$ . (See [Hu2, Lemma 4.20].)

The proof for  $\bar{W}_-$  is similar.  $\square$

**5.7. Holomorphic curves with ends at  $z_\infty$ .** In this subsection we explain how to extend the definitions of the Fredholm and ECH indices to holomorphic curves which have ends at multiples of  $z_\infty$ . The novelty is that the Lagrangian boundary condition is singular at  $z_\infty$  and that the chord over  $z_\infty$  can be used more than once. We will treat in detail the case of a curve  $\bar{u} : \bar{F} \rightarrow \bar{W}$  which is a degree  $l$  multisection of  $\bar{W} \rightarrow \mathbb{R} \times [0, 1]$ ; multisections of  $\bar{W}_+$  and  $\bar{W}_-$  can be treated similarly.

**5.7.1. Data at  $z_\infty^p$ .** We define the data  $\vec{\mathcal{D}}$  at  $z_\infty^p$  as a  $p$ -tuple of matchings

$$\{(i'_1, j'_1) \rightarrow (i_1, j_1), \dots, (i'_p, j'_p) \rightarrow (i_p, j_p)\},$$

where  $i_k, i'_k \in \{1, \dots, 2g\}$ ,  $j_k, j'_k \in \{0, 1\}$  for  $k = 1, \dots, p$  and  $i_k \neq i_l$ ,  $i'_k \neq i'_l$  for  $k \neq l$ . To the data  $\vec{\mathcal{D}}$  we associate its set of *initial points*  $\mathcal{D}^{from} =$

$\{(i'_1, j'_1), \dots, (i'_p, j'_p)\}$  and its set of *terminal points*  $\mathcal{D}^{to} = \{(i_1, j_1), \dots, (i_p, j_p)\}$ , and define  $\mathcal{D} = (\mathcal{D}^{to}, \mathcal{D}^{from})$ .

We write  $\{z_\infty^p(\vec{\mathcal{D}})\} \cup \mathbf{y}$  for the  $2g$ -tuple of points of  $\bar{\mathbf{a}} \cap \bar{h}(\bar{\mathbf{a}})$ , where  $\mathbf{y} \subset S$ ,  $z_\infty$  has multiplicity  $p$ ,  $\vec{\mathcal{D}}$  is the data at  $z_\infty^p$ , and each arc of  $\{\bar{a}_i, \bar{h}(\bar{a}_i)\}_{i=1}^{2g}$  is used once. Here  $z_\infty^p(\vec{\mathcal{D}})$  is viewed as a collection of chords  $z_\infty$  from  $\bar{h}(\bar{a}_{i'_k, j'_k})$  to  $\bar{a}_{i_k, j_k}$ , where  $k = 1, \dots, p$ .

**5.7.2. Multisections.** In this subsection we define multisections  $\bar{u} : (\dot{F}, j) \rightarrow (\overline{W}, \overline{J})$  with irreducible components which branched cover  $\sigma_\infty$ . The notation is complicated by the fact that there may be branch points of  $\bar{\pi}_B \circ \bar{u}$  along  $\partial B$ .

*Notation.* Let  $(F, j)$  be a compact *nodal* Riemann surface, possibly disconnected, with two sets of punctures  $\mathbf{q}^+ = \{q_1^+, \dots, q_k^+\}$  and  $\mathbf{q}^- = \{q_1^-, \dots, q_k^-\}$  on  $\partial F$  and a set  $\mathbf{p} = \{p_1, \dots, p_l\}$  of nodes on  $\partial F$ , such that (i) the nodes  $\mathbf{p}$  are disjoint from  $\mathbf{q}^+$  and  $\mathbf{q}^-$ , (ii) each connected component of  $F$  has nonempty boundary, (iii) on each oriented loop of  $\partial F$  whose orientation agrees with that of  $\partial F$ , there is at least one puncture from each of  $\mathbf{q}^+$  and  $\mathbf{q}^-$ , and (iv) the punctures on  $\mathbf{q}^+$  and  $\mathbf{q}^-$  alternate along each oriented loop of  $\partial F$ . Here, by a *loop or path on  $\partial F$*  we mean a loop or path which is a concatenation of subarcs of  $\partial F$  with endpoints on  $\mathbf{q}^+ \cup \mathbf{q}^- \cup \mathbf{p}$ . We write  $\dot{F} = F - \mathbf{q}^+ - \mathbf{q}^-$ .

Given a holomorphic multisection  $\bar{u} : (\dot{F}, j) \rightarrow (\overline{W}, \overline{J})$ , we write

$$\bar{u} = \bar{u}' \cup \bar{u}'',$$

where  $\bar{u}'$  is a possibly disconnected branched cover of  $\sigma_\infty$  and  $\bar{u}''$  is the union of irreducible components which do not branch cover  $\sigma_\infty$ . We also decompose  $F = F' \sqcup F''$  such that  $\dot{F}'$  is the domain of  $\bar{u}'$ ,  $\dot{F}''$  is the domain of  $\bar{u}''$ , and all the nodes are on  $\partial F'$ . The case of  $\overline{W}_+$  and  $\overline{W}_-$  are analogous.

*The data  $\mathcal{C}$ .* The nodes of  $\partial F'$  correspond to branch points of  $\bar{\pi}_B \circ \bar{u}$  along  $\partial B$ . If we view  $\dot{F}'$  as a branched cover of  $B$  with some branch points along  $\partial B$ , then let  $\dot{F}'_{ext}$  be the branched cover of  $B$  obtained by pushing the boundary branch points to  $int(B)$ .

**Definition 5.7.1** (Data  $\mathcal{C}$ ). An *irreducible component  $d$  of  $\partial \dot{F}'$*  is an oriented path from  $\mathbf{q}^+$  to  $\mathbf{q}^-$  along  $\partial \dot{F}'$ , where the orientation either agrees with that of  $\partial F$  on all the subarcs of  $d$  or is opposite that of  $\partial F$  on all the subarcs of  $d$ . A set  $\Delta = \{d_1, \dots, d_r\}$  is a *decomposition of  $\partial \dot{F}'$  into irreducible components*, if  $\partial \dot{F}' = \cup_{i=1}^r d_i$ ,  $d_i$  is an irreducible component of  $\partial \dot{F}'$ , and  $d_i, d_j, i \neq j$ , intersect only at points of  $\mathbf{q}^+ \cup \mathbf{q}^- \cup \mathbf{p}$ . There is a canonical decomposition  $\Delta$  of  $\partial \dot{F}'$  which corresponds to  $\partial \dot{F}'_{ext}$ . A *data  $\mathcal{C}$*  is a map  $\Delta \rightarrow \{\bar{a}_{i,j}, \bar{h}(\bar{a}_{i,j})\}_{i,j}$ .

In words,  $\bar{u}'$  can be viewed as mapping  $\Delta$  to the set of Lagrangians  $L_{\bar{a}_{i,j}}$  or  $L_{\bar{h}(\bar{a}_{i,j})}$ . Observe that  $\mathcal{C}$  determines the data  $\vec{\mathcal{D}}_+$  and  $\vec{\mathcal{D}}_-$  at the positive and negative ends.

We then make the following definition which agrees with Definition 4.3.1 when  $\mathbf{z}_+ = \mathbf{y}_+$  and  $\mathbf{z}_- = \mathbf{y}_-$ :

**Definition 5.7.2.** Let  $\mathbf{z}_+ = \{z_\infty^{p+}(\vec{\mathcal{D}}_+)\} \cup \mathbf{y}_+$  and  $\mathbf{z}_- = \{z_\infty^{p-}(\vec{\mathcal{D}}_-)\} \cup \mathbf{y}_-$  be  $k$ -tuples of points of  $\bar{a} \cap \bar{h}(\bar{a})$ . A *degree  $k$  multisection of  $(\bar{W}, \bar{J})$  from  $\mathbf{z}_+$  to  $\mathbf{z}_-$*  is a pair  $(\bar{u}, \mathcal{C})$  consisting of a holomorphic map

$$\bar{u} = \bar{u}' \cup \bar{u}'' : (\dot{F} = \dot{F}' \sqcup \dot{F}'', j) \rightarrow (\bar{W}, \bar{J})$$

which is a degree  $k$  multisection of  $\pi_B : \bar{W} \rightarrow B$  and data  $\mathcal{C}$  for  $\bar{u}'$ , and which additionally satisfies the following:

- (1)  $\bar{u}''(\partial \dot{F}'') \subset L_{\bar{a}} \cup L_{\bar{h}(\bar{a})}$ ;
- (2) there is a canonical decomposition  $\Delta$  of  $\partial \dot{F}'$  such that  $\bar{u}$  maps each connected component of  $\partial \dot{F}''$  and each irreducible component of  $\Delta$  to a different  $L_{\bar{a}_i}$  or  $L_{\bar{h}(\bar{a}_i)}$  (here we are using  $\mathcal{C}$  to assign some  $L_{\bar{a}_i}$  or  $L_{\bar{h}(\bar{a}_i)}$  to each irreducible component of  $\Delta$ );
- (3)  $\lim_{w \rightarrow q_i^+} \pi_{\mathbb{R}} \circ \bar{u}(w) = +\infty$  and  $\lim_{w \rightarrow q_i^-} \pi_{\mathbb{R}} \circ \bar{u}(w) = -\infty$ ;
- (4)  $\bar{u}$  converges to a trivial strip over  $[0, 1] \times \mathbf{z}_+$  near  $\mathbf{q}^+$  and to a trivial strip over  $[0, 1] \times \mathbf{z}_-$  near  $\mathbf{q}^-$ ;
- (5) the positive and negative ends of  $\bar{u}$  which limit to  $z_\infty$  are described by  $\vec{\mathcal{D}}_+$  and  $\vec{\mathcal{D}}_-$ ;
- (6) the energy of  $\bar{u}$  is finite.

*Remark 5.7.3.* It will always be assumed that a multisection  $\bar{u}$  of  $\bar{W}$  from  $\{z_\infty^{p+}\} \cup \mathbf{y}_+$  to  $\{z_\infty^{p-}\} \cup \mathbf{y}_-$  comes with data  $\mathcal{C}$ . Indeed, in this paper, such a curve  $\bar{u}$  only appears as the SFT limit of curves  $\bar{u}_i$  without components which branch cover  $\sigma_\infty$  and hence naturally inherits  $\mathcal{C}$  and  $\vec{\mathcal{D}}_\pm$ .

**5.7.3. Multivalued trivializations.** We define the trivialization  $\tau = \tau_{\vec{\mathcal{D}}_+, \vec{\mathcal{D}}_-}$  which will be used in the definition of the Fredholm index of a map  $\bar{u}$  from  $\{z_\infty^{p+}(\vec{\mathcal{D}}_+)\} \cup \mathbf{y}_+$  to  $\{z_\infty^{p-}(\vec{\mathcal{D}}_-)\} \cup \mathbf{y}_-$ . The trivialization  $\tau_{\vec{\mathcal{D}}_+, \vec{\mathcal{D}}_-}$  is *multivalued* near  $z_\infty$  and depends on  $\vec{\mathcal{D}}_+$  and  $\vec{\mathcal{D}}_-$ .

Let  $\tau' = \partial_\rho$  be a trivialization of  $T\bar{S}$  on  $D_\delta^2 - \{z_\infty\} = \{0 < \rho \leq \delta\} \subset \bar{S}$ , where  $\delta > 0$  is small. Let  $\vec{\bar{W}} = [-1, 1] \times [0, 1] \times \bar{S}$ . We extend  $\tau'$  arbitrarily to a trivialization of  $T\bar{S}$  along  $\bar{a}$  and pull it back to  $\tau_{\vec{\mathcal{D}}_+, \vec{\mathcal{D}}_-}$  on  $T\vec{\bar{S}}_{\vec{\bar{W}}}$  along  $[-1, 1] \times \{1\} \times \bar{a}$ ; similarly we extend  $\tau'$  arbitrarily to a trivialization of  $T\bar{S}$  along  $\bar{h}(\bar{a})$  and pull it back to  $\tau_{\vec{\mathcal{D}}_+, \vec{\mathcal{D}}_-}$  on  $T\vec{\bar{S}}_{\vec{\bar{W}}}$  along  $[-1, 1] \times \{0\} \times \bar{h}(\bar{a})$ . Then we extend  $\tau_{\vec{\mathcal{D}}_+, \vec{\mathcal{D}}_-}$  along  $(\{1\} \times [0, 1] \times \mathbf{y}_+) \cup (\{-1\} \times [0, 1] \times \mathbf{y}_-)$  and still denote it by  $\tau_{\vec{\mathcal{D}}_+, \vec{\mathcal{D}}_-}$ . Finally, for each  $(i'_{\pm, k}, j'_{\pm, k}) \rightarrow (i_{\pm, k}, j_{\pm, k})$  in  $\vec{\mathcal{D}}_\pm$ , we choose an extension  $\tau_{\vec{\mathcal{D}}_+, \vec{\mathcal{D}}_-}$  to  $\{\pm 1\} \times [0, 1] \times \{z_\infty\}$  in an arbitrary way.

If  $\vec{\mathcal{D}}_\pm$  are the data induced by a multivalued trivialization  $\tau$ , then we say that  $\tau$  is *compatible* with  $\vec{\mathcal{D}}_\pm$ .

*Remark 5.7.4.* Note that the extensions to  $\bar{a}$  and to  $\bar{h}(\bar{a})$  might conflict, but it does not matter here. In the cobordism cases (i.e., for  $\bar{W}_+$  and  $\bar{W}_-$ ), when  $\bar{a}$  and  $\bar{h}(\bar{a})$  are connected by the Lagrangian  $L_{\bar{a}}^\pm$ , we extend the trivialization  $\tau'$  to  $\bar{a}$  in an

arbitrary manner and then sweep it around using the symplectic connection along the boundary of the cobordism.

**5.7.4. Groomed multivalued trivializations.** Let  $\tau$  be a multivalued trivialization. Let

$$A_\varepsilon = [0, 1] \times \partial D_\varepsilon^2 \subset [0, 1] \times \overline{S},$$

where  $\varepsilon > 0$  is small. We use coordinates  $(t, \phi)$  on

$$A_\varepsilon \simeq ([0, 1] \times [0, 2\pi]) / (t, 0) \sim (t, 2\pi).$$

The branches of the trivialization  $\tau$  along  $[0, 1] \times \{z_\infty\}$  give rise to a family of arcs  $c_k^\pm$  in  $A_\varepsilon$ . Without loss of generality we may assume that  $c_k^\pm$ ,  $k = 1, \dots, p_\pm$ , is linear with respect to the identification of the universal cover of  $A_\varepsilon$  with  $[0, 1] \times \mathbb{R}$ . For each arc  $c_k^\pm$  we denote its initial point by  $p_{k,0}^\pm$  and its terminal point by  $p_{k,1}^\pm$ . We finally define  $P_0^\pm = \{p_{k,0}^\pm\}_{k=1}^{p_\pm}$  and  $P_1^\pm = \{p_{k,1}^\pm\}_{k=1}^{p_\pm}$ . The point  $p_{k,0}^\pm$  belongs to the Lagrangian subarc  $\overline{h}(\overline{a}_{i'_{\pm,k}, j'_{\pm,k}})$  and the point  $p_{k,1}^\pm$  belongs to the Lagrangian subarc  $\overline{a}_{i_{\pm,k}, j_{\pm,k}}$ .

**Definition 5.7.5.** The multivalued trivialization  $\tau$  is *groomed* if the arcs  $\{c_k^*\}_{k=1}^{p_*}$  are pairwise disjoint for both  $* = +$  and  $* = -$ . The collections  $\mathfrak{c}^+ = \{c_k^+\}_{k=1}^{p_+}$  and  $\mathfrak{c}^- = \{c_k^-\}_{k=1}^{p_-}$  are the *groomings* at the positive and negative ends.

Note that every groomed multivalued trivialization induces data  $\vec{\mathcal{D}}_\pm$ , but not every  $\vec{\mathcal{D}}_\pm$  is compatible with a groomed multivalued trivialization.

**5.7.5. Index formulas.** We first give the Fredholm index of a multisection.

**Proposition 5.7.6.** Let  $\overline{u} : \dot{F} \rightarrow \overline{W}$  be a degree  $l$  multisection with data  $\vec{\mathcal{D}}_\pm$  at  $z_\infty$ , and let  $\tau = \tau_{\vec{\mathcal{D}}_+, \vec{\mathcal{D}}_-}$ . Then the Fredholm index of  $\overline{u}$  is given by the formula:

$$(5.7.1) \quad \text{ind}(\overline{u}) = -\chi(F) + l + \mu_\tau(\overline{u}) + 2c_1(\overline{u}^* T\overline{S}, \tau).$$

The proof is a computation in the pullback bundle  $\overline{u}^* T\overline{W}$ , which is completely analogous to that yielding Equation (4.4.4).

Next we discuss the ECH index of a multisection  $(\overline{u}, \mathcal{C})$  with ends

$$\mathbf{z}_+ = \{z_\infty^{p_+}(\vec{\mathcal{D}}_+)\} \cup \mathbf{y}_+, \quad \mathbf{z}_- = \{z_\infty^{p_-}(\vec{\mathcal{D}}_-)\} \cup \mathbf{y}_-.$$

Let  $\tau = \tau_{\vec{\mathcal{D}}_+, \vec{\mathcal{D}}_-}$  be a groomed multivalued trivialization with groomings  $\mathfrak{c}^\pm = \{c_k^\pm\}$ . Let  $Z_{\mathbf{z}_+, \mathbf{z}_-, \tau}$  be the subset of  $\overline{W}$  given by

$$\begin{aligned} Z_{\mathbf{z}_+, \mathbf{z}_-, \tau} = & \check{L}_{\widehat{\mathbf{a}}} \cup \check{L}_{\overline{h}(\widehat{\mathbf{a}})} \cup (\{1\} \times [0, 1] \times \mathbf{y}_+) \cup (\{1\} \times \mathfrak{c}^+) \\ & \cup (\{-1\} \times [0, 1] \times \mathbf{y}_-) \cup (\{-1\} \times \mathfrak{c}^-), \end{aligned}$$

where  $\check{L}_{\widehat{\mathbf{a}}} = [-1, 1] \times \{1\} \times \widehat{\mathbf{a}}$  and  $\check{L}_{\overline{h}(\widehat{\mathbf{a}})} = [-1, 1] \times \{0\} \times \overline{h}(\widehat{\mathbf{a}})$ .

**Remark 5.7.7.** Note that  $[-1, 1] \times \{0, 1\} \times \{z_\infty\}$  is disjoint from  $\check{L}_{\widehat{\mathbf{a}}}$  and  $\check{L}_{\overline{h}(\widehat{\mathbf{a}})}$ .

$\tau$ -trivial representative. Define  $\pi_2(\mathbf{z}_+, \mathbf{z}_-, \tau) \subset H_2(\overline{W}, Z_{\mathbf{z}_+, \mathbf{z}_-, \tau})$  in analogy to the definition of  $\pi_2(\mathbf{y}, \mathbf{y}')$  in Section 4. We now describe the construction of a  $\tau$ -trivial representative  $\check{C} \subset \overline{W}$  of the class  $[\overline{u}]$  in  $\pi_2(\mathbf{z}_+, \mathbf{z}_-, \tau)$ :

*Step 1.* Replace  $\overline{u}' : \dot{F}' \rightarrow \overline{W}$  by  $\overline{u}'_{ext} : \dot{F}'_{ext} \rightarrow \overline{W}$  so that there are no nodes along  $\partial \dot{F}'_{ext}$  and each component of  $\partial \dot{F}'_{ext}$  is mapped to some  $L_{\overline{a}_{i,j}}$  or  $L_{\overline{h}(\overline{a}_{i,j})}$  in a manner consistent with the data  $\mathcal{C} : \Delta \rightarrow \{\overline{a}_{i,j}, \overline{h}(\overline{a}_{i,j})\}_{i,j}$ .

*Step 2.* Compactify the ends of  $\overline{u}_{ext} = \overline{u}'_{ext} \cup \overline{u}''$  to  $\check{\overline{u}}_{ext}$  as in Section 4.3.

*Step 3.* Perturb  $\check{\overline{u}}_{ext}$  so that the resulting representative  $\check{C}$  is immersed,  $\partial \check{C} \subset Z_{\mathbf{z}_+, \mathbf{z}_-, \tau}$ , each component of  $\partial \check{C} \cap \{t = 0, 1\}$  is mapped to  $\check{L}_{\widehat{a}_{i,j}}$  or  $\check{L}_{\widehat{h}(\widehat{a}_{i,j})}$  as specified by  $\mathcal{C}$ , and  $\check{C}$  satisfies

$$\pi(\check{C}|_{s=\pm 1}) \cap ([0, 1] \times \text{int}(D^2)) = \mathbf{c}^\pm \subset A_\varepsilon,$$

and the conditions in Definition 4.5.2 at all the other ends. Then resolve the self-intersections to make  $\check{C}$  embedded. Here

$$\pi : [-1, 1] \times [0, 1] \times \overline{S} \rightarrow [0, 1] \times \overline{S}$$

is the projection onto the second and third factors.

*The quadratic form  $Q_\tau$ .* Let  $\check{C}$  be a  $\tau$ -trivial representative of  $\overline{u}$ . Let  $c_k^\pm(\delta)$  be the  $\phi = \delta$  translate of  $c_k^\pm$ , where  $\delta > 0$  is small. We then take a pushoff  $\check{C}'$  of  $\check{C}$  such that  $\partial \check{C}$  is pushed in the direction of  $J\tau$  and

$$\pi(\check{C}'|_{s=\pm 1}) \cap ([0, 1] \times \text{int}(D^2)) = \cup_{k=1}^{p_\pm} c_k^\pm(\delta) \subset A_\varepsilon.$$

Then  $Q_\tau(A) = \langle \check{C}, \check{C}' \rangle$  as in Definition 4.5.5.

We remark that, in passing from  $\overline{u}$  to  $[\overline{u}]$ , we only record  $\mathcal{D}_\pm$  (and not  $\overrightarrow{\mathcal{D}}_\pm$ ), because the matching is not a homological invariant.

**Definition 5.7.8** (ECH index). Given  $A \in \pi_2(\mathbf{z}_+, \mathbf{z}_-, \tau)$  and a groomed multivalued trivialization  $\tau$  compatible with  $\mathcal{D}_+$  and  $\mathcal{D}_-$ , we define

$$(5.7.2) \quad I_\tau(A) = Q_\tau(A) + \tilde{\mu}_\tau(\partial A) + c_1(T\overline{S}|_A, \tau),$$

where  $\tilde{\mu}_\tau(\partial A)$  will be given in Section 5.7.6.

Sometimes we will write  $I_\tau(u)$  to mean  $I_\tau(A)$ , where  $A$  is the relative homology class defined by  $u$ .

**5.7.6. Definition of  $\tilde{\mu}_\tau(\partial A)$ .** We first define  $\mu_\tau(\partial A)$ . The groomed multivalued trivialization  $\tau$  determines the matchings  $\mathcal{D}_\pm^{from} \rightarrow \mathcal{D}_\pm^{to}$ , and we pick a cycle  $\zeta$  which represents  $\partial A$  and respects the matchings along  $\{\pm 1\} \times [0, 1] \times \{z_\infty\}$ . The pullback bundle  $\zeta^* T\overline{S}$  is trivialized by the pullback of  $\tau$ . We now define a multivalued real rank one subbundle  $\mathcal{L}_0$  of  $T\overline{S}$  along  $Z_{\overline{\mathbf{a}}, \overline{h}(\overline{\mathbf{a}})}$  by setting  $\mathcal{L}_0 = T\check{L}_{\overline{\mathbf{a}}} \cap T\overline{S}$  on  $\check{L}_{\overline{\mathbf{a}}}$  and  $T\check{L}_{\overline{h}(\overline{\mathbf{a}})} \cap T\overline{S}$  on  $\check{L}_{\overline{h}(\overline{\mathbf{a}})}$  and extending  $\mathcal{L}_0$  across  $\{\pm 1\} \times [0, 1] \times \mathbf{z}_\pm$  by rotating in the counterclockwise direction from  $T\overline{h}(\overline{\mathbf{a}})$  to  $T\overline{\mathbf{a}}$  in  $T\overline{S}$



by the minimum amount possible. We then define  $\mu_\tau(\partial A)$  as the Maslov index of  $\mathcal{L} = \zeta^* \mathcal{L}_0$  with respect to  $\tau$ .

Now we define the corrections to add to  $\mu_\tau(\partial A)$  to obtain  $\tilde{\mu}_\tau(\partial A)$ . Let  $\tau$  be a groomed multivalued trivialization which is compatible with  $(\mathcal{D}_+, \mathcal{D}_-)$  and which induces the groomings  $\mathfrak{c}^+ = \{c_k^+\}_{k=1}^{p_+}$  and  $\mathfrak{c}^- = \{c_k^-\}_{k=1}^{p_-}$ .

**Definition 5.7.9.** Given a grooming  $\mathfrak{c} = \{c_k\}_{k=1}^p$  from  $P_0$  to  $P_1$ , its *winding number* is given by:

$$(5.7.3) \quad w(\mathfrak{c}) = \sum_k w(c_k), \quad w(c_k) = \langle c_k, \{\phi = \pi\} \rangle,$$

where the arcs  $c_k$  and  $\{\phi = \pi\}$  are oriented from  $t = 0$  to  $t = 1$  and  $A_\varepsilon$  is oriented by  $(\partial_\phi, \partial_t)$ .

*Remark 5.7.10.* The grooming  $\mathfrak{c}$  is determined by its endpoints  $P_0$  and  $P_1$  and its winding number  $w(\mathfrak{c})$ .

Let  $w(\mathfrak{c}) = q$ . Let  $c_k^b$  be the linear arc in  $A_\varepsilon$  which is disjoint from  $\{\phi = \pi\}$  and has the same endpoints as  $c_k$ . (Note that the collection  $\{c_k^b\}$  is usually not groomed.) Then we define  $\alpha_q$  as the number of arcs in  $\{c_k^b\}$  whose  $\phi$ -coordinate decreases as  $t$  increases (in the universal cover). The number  $\alpha_q$  only depends on  $q$  (modulo  $p$ ), through the bijection  $P_0 \xrightarrow{\sim} P_1$ .

We finally define  $\tilde{\mu}_\tau(\partial A)$  as follows: Let  $q_+ = w(\mathfrak{c}^+)$  and  $q_- = w(\mathfrak{c}^-)$ . Then we define the “discrepancies”  $d^\pm$  at the positive and negative ends as:

$$(5.7.4) \quad d^\pm = -(\alpha_{q_\pm} - \alpha_0) - q_\pm(p_\pm - 1)$$

and we set:

$$(5.7.5) \quad \tilde{\mu}_\tau(\partial A) = \mu_\tau(\partial A) + d^+ - d^-.$$

(This means that we are using  $q_- = 0$  as the reference point.) The discrepancy  $d^+ - d^-$  was (somewhat artificially) added to make Lemma 5.7.12 hold.

*Remark 5.7.11.* Let us view  $P_0$  and  $P_1$  as points on  $(-\pi, \pi)$ . In the special case where the points of  $P_0$  and  $P_1$  alternate along  $(-\pi, \pi)$ , we can write:

$$(5.7.6) \quad d^* = \begin{cases} -p_*(q_* - \lfloor \frac{q_*}{p_*} \rfloor), & \text{if } \min_{x \in P_0} x < \min_{x \in P_1} x; \\ -p_*(q_* - \lceil \frac{q_*}{p_*} \rceil), & \text{if } \min_{x \in P_0} x > \min_{x \in P_1} x. \end{cases}$$

Here  $\lfloor x \rfloor$  is the greatest integer  $\leq x$  and  $\lceil x \rceil$  is the smallest integer  $\geq x$ .

*5.7.7. ECH indices of branched covers of sections at infinity.*

**Lemma 5.7.12.** *If  $A \in \pi_2(\{z_\infty^p(\vec{\mathcal{D}}_+), \{z_\infty^p(\vec{\mathcal{D}}_-)\}, \tau)$  is the relative homology class of a  $p$ -fold branched cover of  $\sigma_\infty$  (with possibly empty branch locus), then  $I_\tau(A) = 0$  for all groomed multivalued trivializations  $\tau$  compatible with  $\mathcal{D}_+$  and  $\mathcal{D}_-$ .*

*Proof.* Let  $\mathfrak{c}^+ = \{c_k^+\}$  and  $\mathfrak{c}^- = \{c_k^-\}$  be the groomings of  $\tau$  and let  $q_\pm = w(\mathfrak{c}^\pm)$ .

In order to compute  $Q_\tau(A)$ , we choose a  $\tau$ -trivial representative  $\check{C}$  of  $A$  such that:

- $\check{C} \cap ([0, 1] \times [0, 1] \times \overline{S}) = [0, 1] \times \mathfrak{c}^+$ ;
- $\check{C} \cap ((-1, 0) \times [0, 1] \times \overline{S}) \subset (-1, 0) \times [0, 1] \times \text{int}(D_\varepsilon^2)$ ;
- $\check{C} \cap (\{-1\} \times [0, 1] \times \overline{S}) = \{-1\} \times \mathfrak{c}^-$ ;

and a representative  $\check{C}'$  such that:

- $\check{C}' \cap (\{1\} \times [0, 1] \times \overline{S}) = \{1\} \times \mathfrak{c}^+(\delta)$ ;
- $\check{C}' \cap ((0, 1) \times [0, 1] \times \overline{S}) \subset (0, 1) \times [0, 1] \times \text{int}(D_\varepsilon^2)$ ;
- $\check{C}' \cap ([-1, 0] \times [0, 1] \times \overline{S}) = [-1, 0] \times \mathfrak{c}^-(\delta)$ .

Since all the intersections between  $\check{C}$  and  $\check{C}'$  are contained in the level  $s = 0$ ,

$$Q_\tau(A) = \mathfrak{c}^+ \cdot \mathfrak{c}^-(\delta) = p(q_+ - q_-).$$

Next we claim that

$$\mu_\tau(\partial A) = (\alpha_{q_+} - 2q_+) - (\alpha_{q_-} - 2q_-).$$

Indeed, given the end which corresponds to a strand  $c_k^\pm$ , the Maslov index of the end is given by  $-2w(c_k^\pm)$  if the endpoints of  $c_k^\pm$  satisfy  $0 < p_{k,0}^\pm < p_{k,1}^\pm < 2\pi$ , and is given by  $1 - 2w(c_k^\pm)$  if  $0 < p_{k,1}^\pm < p_{k,0}^\pm < 2\pi$ . On the other hand, the number of strands for which  $p_{k,0}^\pm > p_{k,1}^\pm$  holds is exactly  $\alpha_q$ . The claim follows.

Hence,

$$\begin{aligned} \tilde{\mu}_\tau(\partial A) &= (\alpha_{q_+} - 2q_+) - (\alpha_{q_+} - \alpha_0) - q_+(p - 1) \\ &\quad - (\alpha_{q_-} - 2q_-) + (\alpha_{q_-} - \alpha_0) + q_-(p - 1) \\ &= -(q_+ - q_-)(p + 1). \end{aligned}$$

We also have  $c_1(T\overline{S}|_A, \tau) = q_+ - q_-$ . Putting everything together, we obtain:

$$I_\tau(A) = p(q_+ - q_-) - (q_+ - q_-)(p + 1) + (q_+ - q_-) = 0.$$

This proves the lemma.  $\square$

We also state the following lemmas without proof:

**Lemma 5.7.13.** *If  $\overline{u} : \dot{F} \rightarrow \overline{W}_-$  is a degree  $p \leq 2g$  multisection which branch covers  $\sigma_\infty^-$  with possibly empty branch locus, then  $I(\overline{u}) = 0$ .*

**Lemma 5.7.14.** *If  $\overline{u} : \dot{F} \rightarrow \overline{W}_- - \text{int}(W_-)$  is a degree  $p \leq 2g$  multisection with positive ends at  $\delta_0$  with total multiplicity  $p$  and negative ends at a  $p$ -element subset of  $\{x_1, \dots, x_{2g}, x'_1, \dots, x'_{2g}\}$ , then  $I(\overline{u}) = p$ .*

**5.7.8. Additivity of indices and independence of the trivialization.** The Fredholm and the ECH index are additive with respect to concatenation. The proofs of the following lemmas are straightforward.

**Lemma 5.7.15.** *Let  $\overline{u}_1$  be a multisection with negative end  $\{z_\infty^p(\overrightarrow{\mathcal{D}})\} \cup \mathbf{y}$ , and let  $\overline{u}_2$  be a multisection with positive end  $\{z_\infty^p(\overrightarrow{\mathcal{D}})\} \cup \mathbf{y}$ . If  $\overline{u}_1 \# \overline{u}_2$  is the multisection obtained by gluing  $\overline{u}_1$  and  $\overline{u}_2$  along their common end, then*

$$\text{ind}(\overline{u}_1 \# \overline{u}_2) = \text{ind}(\overline{u}_1) + \text{ind}(\overline{u}_2).$$

**Lemma 5.7.16.** *Let  $\tau_2$  and  $\tau_1$  be groomed multivalued trivializations compatible with  $(\mathcal{D}_2, \mathcal{D}_1)$  and  $(\mathcal{D}_1, \mathcal{D}_0)$ , respectively, and let  $\tau$  be obtained by concatenating  $\tau_2$  and  $\tau_1$ . Given relative homology classes*

$$A_2 \in \pi_2(\{z_\infty^{p_2}(\vec{\mathcal{D}}_2)\} \cup \mathbf{y}_2, \{z_\infty^{p_1}(\vec{\mathcal{D}}_1)\} \cup \mathbf{y}_1, \tau_2),$$

$$A_1 \in \pi_2(\{z_\infty^{p_1}(\vec{\mathcal{D}}_1)\} \cup \mathbf{y}_1, \{z_\infty^{p_0}(\vec{\mathcal{D}}_0)\} \cup \mathbf{y}_0, \tau_1),$$

*we can form the concatenation*

$$A_2 \# A_1 \in \pi_2(\{z_\infty^{p_2}(\vec{\mathcal{D}}_2)\} \cup \mathbf{y}_2, \{z_\infty^{p_0}(\vec{\mathcal{D}}_0)\} \cup \mathbf{y}_0, \tau).$$

*Then we have:*

$$I_\tau(A_2 \# A_1) = I_{\tau_2}(A_2) + I_{\tau_1}(A_1).$$

In view of the following lemma, we can suppress  $\tau$  from  $I_\tau$ .

**Lemma 5.7.17.**  *$I_\tau(A)$  is independent of the choice of groomed multivalued trivialization  $\tau$ .*

*Proof.* Let  $\tau$  and  $\tau'$  be two groomed multivalued trivializations adapted to the same data  $(\mathcal{D}_+, \mathcal{D}_-)$ . It suffices to consider the particular cases when  $\tau$  and  $\tau'$  differ only either at some  $y_i$  or at  $z_\infty$ . In the first case, the argument is analogous to the proof of Lemma 4.5.6. In the second case, we can glue branched covers of  $\sigma_\infty$  to switch groomings. Then the statement follows from Lemma 5.7.12 and the additivity of the ECH index.  $\square$

**5.7.9. The ECH index inequality.** Let  $\bar{u}$  be a degree  $l$  multisection of  $\overline{W}$  from  $\mathbf{z}_+ = \{z_\infty^{p_+}(\vec{\mathcal{D}}_+)\} \cup \mathbf{y}_+$  to  $\mathbf{z}_- = \{z_\infty^{p_-}(\vec{\mathcal{D}}_-)\} \cup \mathbf{y}_-$  such that  $\bar{u} = \bar{u}'$ . We define  $c_k^\pm$ ,  $k = 1, \dots, p_\pm$  as the intersections of  $A_\varepsilon$  with the  $\pi$ -projections of the  $\pm$  ends of  $\bar{u}$  which limit to  $z_\infty$ . Here  $\varepsilon > 0$  is small and depends on  $\bar{u}$ , and the map  $\pi$  projects out the  $s$ -direction. Let  $\mathbf{c}^\pm = \{c_k^\pm\}_{k=1, \dots, p_\pm}$  and let  $P_0^\pm$  (resp.  $P_1^\pm$ ) be the set of the initial (resp. terminal) points of the arcs  $c_k^\pm$ .

**Lemma 5.7.18.** *If  $\tau$  is a groomed multivalued trivialization which is compatible with  $\mathbf{c}^\pm$  and  $A \in \pi_2(\mathbf{z}_+, \mathbf{z}_-, \tau)$  is the relative homology class of  $\bar{u}$ , then*

$$(5.7.7) \quad I(A) \geq \text{ind}(\bar{u}) + (d^+ - d^-).$$

*In particular, if the points of  $P_0^*$  and  $P_1^*$  alternate along  $(0, 2\pi)$  for both  $*$  = + and -, then  $I(A) \geq \text{ind}(\bar{u})$ .*

*Proof.* If we use the groomed trivialization  $\tau$  corresponding to  $\mathbf{c}^\pm$  to compute both the Fredholm and ECH indices, then the proof of the relative adjunction formula (Lemma 4.5.9) goes through unmodified. Hence Equation (5.7.1) and Lemma 4.5.9 imply:

$$\begin{aligned} \text{ind}(\bar{u}) &= -\chi(F) + l + \mu_\tau(\bar{u}) + 2c_1(\bar{u}^* T \overline{S}, \tau) \\ &= Q_\tau(A) + \mu_\tau(\bar{u}) + c_1(\bar{u}^* T \overline{S}, \tau) - 2\delta(\bar{u}). \end{aligned}$$

Since  $\tau$  is groomed, we have  $\mu_\tau(\bar{u}) = \mu_\tau(\partial A)$ . Comparing with the definition of  $I(\bar{u})$ , we have:

$$(5.7.8) \quad \begin{aligned} I(\bar{u}) &= \text{ind}(\bar{u}) + (\tilde{\mu}_\tau(\partial A) - \mu_\tau(\partial A)) + 2\delta(\bar{u}) \\ &= \text{ind}(\bar{u}) + (d^+ - d^-) + 2\delta(\bar{u}). \end{aligned}$$

Equation (5.7.7) follows from observing that  $\delta(\bar{u}) \geq 0$  since  $\bar{u}$  is  $J$ -holomorphic.

Next suppose that the points of  $P_0^*$  and  $P_1^*$  alternate along  $(0, 2\pi)$  for both  $*$  = + and  $-$ . We claim that  $d^+ \geq 0$ . Let  $\pi : \mathbb{R} \times [0, 1] \times \bar{S} \rightarrow [0, 1] \times \bar{S}$  be the projection onto the second and third factors. By the positivity of intersections,  $\pi(\bar{u})$  is positively transverse to the Hamiltonian vector field  $\partial_t$ ; hence  $q_+ \leq 0$ . By Equation (5.7.6), if  $q_+ \leq 0$ , then  $d^+ \geq 0$  (in both cases).  $d^- \leq 0$  is proved similarly.  $\square$

**5.7.10. ECH index calculation.** In this subsection we compute the ECH index of a multisection of  $\bar{W}$  which is the disjoint union of a branched cover of the section at infinity  $\sigma_\infty$  and a curve which limits to a multiple of  $z_\infty$ . This calculation will be used in Section II.3.7.2.

Let  $\bar{u} = \bar{u}' \cup \bar{u}''$  be a degree  $p_1 + p_2 + l$  multisection of  $\bar{W}$ , where  $\deg \bar{u}' = p_1$ ,  $\bar{u}''$  is a multisection from  $\mathbf{y}$  to  $\{z_\infty^{p_2}(\bar{\mathcal{D}}_2)\} \cup \mathbf{y}'$  and  $l$  is the cardinality of  $\mathbf{y}'$ . Suppose that the data  $\bar{\mathcal{D}}_1$  and  $\bar{\mathcal{D}}_2$  for  $\bar{u}'$  and  $\bar{u}''$  at the negative ends  $z_\infty^{p_1}$  and  $z_\infty^{p_2}$  satisfy:

- (D1)  $\bar{\mathcal{D}}_1 = \{(i_1, j_1) \rightarrow (i_1, j_1), \dots, (i_{p_1}, j_{p_1}) \rightarrow (i_{p_1}, j_{p_1})\}$ ;
- (D2)  $\bar{\mathcal{D}}_2 = \{(i'_1, j'_1) \rightarrow (i''_1, j''_1), \dots, (i'_{p_2}, j'_{p_2}) \rightarrow (i''_{p_2}, j''_{p_2})\}$ ;
- (D3)  $\mathcal{D}_1^{from} = \mathcal{D}_1^{to}$  and the sets  $\mathcal{D}_2^{from}$  and  $\mathcal{D}_2^{to}$  are disjoint from  $\mathcal{D}_1^{from} = \mathcal{D}_1^{to}$ .

Let  $\mathcal{E}_{-,k}$  be the end of  $\bar{u}''$  at  $z_\infty$  corresponding to  $(i'_k, j'_k) \rightarrow (i''_k, j''_k)$ .

Let  $\mathfrak{c}_1^- \subset A_{\varepsilon/2} = \partial D_{\varepsilon/2}^2 \times [0, 1]^8$  and  $\mathfrak{c}_2^- \subset A_\varepsilon = \partial D_\varepsilon^2 \times [0, 1]$  be groomings which correspond to  $\bar{\mathcal{D}}_1$  and  $\bar{\mathcal{D}}_2$  and satisfy the following:

- (G1)  $\mathfrak{c}_1^-$  has winding number  $q_1 := w(\mathfrak{c}_1^-) = 0$ ;
- (G2) the intersection  $\pi(\cup_{i=1}^{p_2} \mathcal{E}_{-,k}) \cap A_\varepsilon$  is groomed;
- (G3)  $\mathfrak{c}_2^- = \pi(\cup_{i=1}^{p_2} \mathcal{E}_{-,k}) \cap A_\varepsilon$  and  $q_2 := w(\mathfrak{c}_2^-) = 0$  or  $1$ ;
- (G4) the points of  $P_0^2$  and  $P_1^2$  alternate along  $(0, 2\pi)$ , i.e., the projection of  $\mathfrak{c}_2^-$  to  $\partial D_\varepsilon^2$  is injective.

Here  $P_0^i$  (resp.  $P_1^i$ ) is the set of initial (resp. terminal) points of  $\mathfrak{c}_i^-$ .

Let  $\pi$  be the projection of  $\bar{W} = [-1, 1] \times [0, 1] \times \bar{S}$  to  $[0, 1] \times \bar{S}$  and let  $\check{C}'$  and  $\check{C}''$  be representatives of  $\bar{u}'$  and  $\bar{u}''$  in  $\bar{W}$  such that  $\pi(\check{C}'|_{s=-1}) = \mathfrak{c}_1^-$  and  $\pi(\check{C}''|_{s=-1}) = \mathfrak{c}_2^-$ .

Let  $\pi_\rho : [\frac{\varepsilon}{2}, \varepsilon] \times \partial D^2 \times [0, 1] \rightarrow \partial D^2 \times [0, 1]$  be the projection along the  $\rho$ -direction. Let  $\mathfrak{w}$  be the signed number of crossings of  $\pi_\rho(\mathfrak{c}_1^- \cup \mathfrak{c}_2^-)$ , i.e., the writhe. Observe that all the crossings of  $\pi_\rho(\mathfrak{c}_1^- \cup \mathfrak{c}_2^-)$  are positive as a consequence of (G1)–(G3) and  $p_1 \geq \mathfrak{w}$  as a consequence of (G4).

<sup>8</sup>Here we are writing  $A_{\varepsilon/2} = \partial D_{\varepsilon/2}^2 \times [0, 1]$  instead of  $[0, 1] \times \partial D_{\varepsilon/2}^2$  to indicate the orientation of  $A_{\varepsilon/2}$ .

**Lemma 5.7.19.** *Let  $\bar{u} = \bar{u}' \cup \bar{u}''$  be a degree  $p_1 + p_2 + l$  multisection of  $\overline{W}$ , where  $\deg \bar{u}' = p_1$ ,  $\bar{u}''$  is a multisection from  $\mathbf{y}$  to  $\{z_\infty^{p_2}(\overrightarrow{\mathcal{D}}_2)\} \cup \mathbf{y}'$  and  $l$  is the cardinality of  $\mathbf{y}'$ . If the data  $\overrightarrow{\mathcal{D}}_1, \overrightarrow{\mathcal{D}}_2$  at the negative end satisfy  $(D_1)-(D_3)$  and  $(G_1)-(G_4)$ , then*

$$(5.7.9) \quad I(\bar{u}) \geq I(\bar{u}') + I(\bar{u}'') + \begin{cases} 2\mathfrak{w}, & \text{if } q_2 = 0; \\ p_1 + \mathfrak{w}, & \text{if } q_2 = 1. \end{cases}$$

If  $\bar{u}' \cap \bar{u}'' = \emptyset$  in addition, then equality holds.

*Proof.* It suffices to prove the equality, assuming  $\bar{u}' \cap \bar{u}'' = \emptyset$ , since the extra intersections contribute positively towards the ECH index. The representatives  $\check{C}'$  and  $\check{C}''$  can be taken to be disjoint; in fact we can assume that  $\check{C}''$  is disjoint from the  $\frac{\varepsilon}{2}$ -neighborhood  $\mathbb{R} \times [0, 1] \times D_{\varepsilon/2}^2$  of  $\sigma_\infty$ .

If  $q_2 = 0$ , then we resolve the positive crossings of  $\pi_\rho(\mathfrak{c}_1^- \cup \mathfrak{c}_2^-)$ . This is equivalent to appending a union of disks  $\check{D} \subset [-2, -1] \times [0, 1] \times \overline{S}$  to  $\check{C}' \cup \check{C}''$  such that  $\check{D}|_{s=-1} = (\check{C}' \cup \check{C}'')|_{s=-1}$  and  $\pi(\check{D}|_{s=-2})^9$  is a grooming on  $A_\varepsilon$  which satisfies  $w(\pi(\check{D}|_{s=-2})) = 0$ . A quick calculation shows that  $\check{D}$  contributes  $\mathfrak{w}$  to  $Q$ , 0 to  $c_1$ , and  $\mathfrak{w}$  to  $\mu$ . Since the discrepancy for  $\pi(\check{D}|_{s=-2})$  is zero, Equation (5.7.9) follows for  $q_2 = 0$ .

If  $q_2 = 1$ , then we first switch the crossings of  $\pi_\rho(\mathfrak{c}_1^- \cup \mathfrak{c}_2^-)$  by appending a union of disks  $\check{D}_{-1} \subset [-2, -1] \times [0, 1] \times \overline{S}$  such that  $\pi(\check{D}_{-1}|_{s=-2})$  consists of  $\mathfrak{c}_1^-$  on  $A_\varepsilon$  and  $\mathfrak{c}_2^-$  on  $A_{\varepsilon/2}$ . This contributes  $2\mathfrak{w}$  to  $Q$  and 0 to  $c_1$  and  $\mu$ . Next append  $\check{D}_{-2} \subset [-3, -2] \times [0, 1] \times \overline{S}$  so that  $\pi(\check{D}_{-2}|_{s=-3})$  consists of  $\mathfrak{c}_1^-$  on  $A_\varepsilon$  and  $\mathfrak{d}_2^-$  on  $A_{\varepsilon/2}$ , where  $\mathfrak{d}_2^-$  is groomed and  $w(\mathfrak{d}_2^-) = 0$ . The surface  $\check{D}_{-2}$  is similar to that used in the proof of Lemma 5.7.12 and has zero ECH index. Finally, we resolve the positive crossings of  $\pi_\rho(\mathfrak{c}_1^- \cup \mathfrak{d}_2^-)$  by appending  $\check{D}_{-3} \subset [-4, -3] \times [0, 1] \times \overline{S}$ . Since there are  $p_1 - \mathfrak{w}$  crossings,  $\check{D}_{-3}$  contributes  $p_1 - \mathfrak{w}$  to  $Q$ , 0 to  $c_1$ , and 0 to  $\mu$ . Equation (5.7.9) then follows for  $q_2 = 1$ .  $\square$

Next we consider the variant where the multiple of  $z_\infty$  is at the positive end. Let  $\bar{u} = \bar{u}' \cup \bar{u}''$  be a degree  $p_1 + p_2 + l$  multisection of  $\overline{W}$ , where  $\deg \bar{u}' = p_1$ ,  $\bar{u}''$  is a multisection from  $\{z_\infty^{p_2}(\overrightarrow{\mathcal{D}}_2)\} \cup \mathbf{y}'$  to  $\mathbf{y}$ , and  $l$  is the cardinality of  $\mathbf{y}'$ . We use the same notation as above, with  $-$  replaced by  $+$ .

We make the following assumptions:

- (G'\_1)  $\mathfrak{c}_1^+$  has winding number  $q_1 := w(\mathfrak{c}_1^+) = 0$ ;
- (G'\_2) the intersection  $\pi(\cup_{i=1}^{p_2} \mathcal{E}_{+,k}) \cap A_\varepsilon$  is groomed;
- (G'\_3)  $\mathfrak{c}_2^+ = \pi(\cup_{i=1}^{p_2} \mathcal{E}_{+,k}) \cap A_\varepsilon$  and  $q_2 := w(\mathfrak{c}_2^+) = 0$  or  $-1$ ; and
- (G'\_4) the projection of  $\mathfrak{c}_2^+$  to  $\partial D_\varepsilon^2$  is injective except on  $\kappa \geq 0$  short intervals of  $\partial D_\varepsilon^2$  which correspond to thin sectors of type  $\mathfrak{S}(\bar{a}_{i,j}, \bar{h}(\bar{a}_{i,j}))$ .

Let  $\mathfrak{w}$  be the signed number of crossings of  $\pi_\rho(\mathfrak{c}_1^+ \cup \mathfrak{c}_2^+)$ . In this case, all the crossings of  $\pi_\rho(\mathfrak{c}_1^+ \cup \mathfrak{c}_2^+)$  are negative. The analog of Lemma 5.7.19, stated without proof, is:

<sup>9</sup>Here we are taking  $\pi$  to be the projection of the appropriate space to  $[0, 1] \times \overline{S}$ .

**Lemma 5.7.20.** *Let  $\bar{u} = \bar{u}' \cup \bar{u}''$  be a degree  $p_1 + p_2 + l$  multisection of  $\bar{W}$ , where  $\deg \bar{u}' = p_1$ ,  $\bar{u}''$  is a multisection from  $\{z_\infty^{p_2}(\bar{\mathcal{D}}_2)\} \cup \mathbf{y}'$  to  $\mathbf{y}$ , and  $l$  is the cardinality of  $\mathbf{y}'$ . If the data  $\bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2$  at the positive end satisfy  $(D_1)-(D_3)$  and  $(G'_1)-(G'_4)$ , then*

$$(5.7.10) \quad I(\bar{u}) \geq I(\bar{u}') + I(\bar{u}'') + \begin{cases} -\mathfrak{w}, & \text{if } q_2 = 0; \\ -2\mathfrak{w}, & \text{if } q_2 = -1. \end{cases}$$

*If  $\bar{u}' \cap \bar{u}'' = \emptyset$  in addition, then equality holds.*

We also compute the discrepancy  $d_+$  of  $\mathfrak{c}_2^+$  when  $q_2 = -1$ . Since  $\alpha_{-1} = p_2 - 1$  and  $\alpha_0 = \kappa$ ,

$$(5.7.11) \quad \begin{aligned} d_+ &= -(\alpha_{-1} - \alpha_0) - q_2(p_2 - 1) \\ &= -((p_2 - 1) - \kappa) - (-1)(p_2 - 1) = \kappa. \end{aligned}$$

By Lemma 5.7.18,

$$(5.7.12) \quad I(\bar{u}'') \geq \text{ind}(\bar{u}'') + \kappa.$$

**5.7.11. Extended moduli spaces.** We now describe the extended moduli spaces which involve multiples of  $z_\infty$  at the ends. Details will be given for  $\bar{W}$ ; the  $\bar{W}_+$  and  $\bar{W}_-$  cases are analogous.

Let  $\mathcal{M} = \mathcal{M}_{\bar{J}}(\mathbf{z}_+, \mathbf{z}_-)$  be the moduli space of multisections of  $(\bar{W}, \bar{J})$  from  $\mathbf{z}_+ = \{z_\infty^{p_+}(\bar{\mathcal{D}}_+)\} \cup \mathbf{y}_+$  to  $\mathbf{z}_- = \{z_\infty^{p_-}(\bar{\mathcal{D}}_-)\} \cup \mathbf{y}_-$ . Let  $\dagger$  be the modifier “ $\bar{u}' = \emptyset$ ”. We now describe an enlargement of the moduli space  $\mathcal{M}^\dagger$ .

Recall that  $\bar{a}_{i,j} \subset D^2$  is of the form  $\{-1 < \rho \leq 1, \phi = \phi_{i,j}\}$  for some constant  $\phi_{i,j}$ .

**Definition 5.7.21.** An *extended  $\bar{W}$ -curve*  $\bar{u}$  from  $\mathbf{z}_+$  to  $\mathbf{z}_-$  is a multisection of  $(\bar{W}, \bar{J})$  which satisfies the conditions of Definition 5.7.2 with  $\bar{F}' = \emptyset$  and (1) and (2) replaced by the following:

- (1') There exists positive and negative ends  $\mathcal{E}_{+,i}$  and  $\mathcal{E}_{-,i}$  of  $\bar{F}$  that limit to  $z_\infty$  such that:
  - $\bar{u}(\partial \bar{F} - \cup_i \mathcal{E}_{+,i} - \cup_i \mathcal{E}_{-,i}) \subset L_{\hat{\mathbf{a}}} \cup L_{\bar{h}(\hat{\mathbf{a}})}$ ;
  - the two components of  $\bar{u}(\partial \bar{F} \cap \mathcal{E}_{+,i})$  (resp.  $\bar{u}(\partial \bar{F} \cap \mathcal{E}_{-,i})$ ) are subsets of  $L_{\bar{a}_{i_k, j_k}}$  and  $L_{\bar{h}(\bar{a}_{i'_k, j'_k})}$ , where  $(i'_k, j'_k) \rightarrow (i_k, j_k) \in \bar{\mathcal{D}}_+$  (resp.  $\in \bar{\mathcal{D}}_-$ ).
- (2')  $\bar{u}$  maps each connected component of  $\partial \bar{F} - \cup_i \mathcal{E}_{+,i} - \cup_i \mathcal{E}_{-,i}$  to a different  $L_{\hat{a}_i}$  or  $L_{\bar{h}(\hat{a}_i)}$ .

The moduli space of extended  $\bar{W}$ -curves from  $\mathbf{z}_+$  to  $\mathbf{z}_-$  is denoted by  $\mathcal{M}_{\bar{J}}^{\dagger, ext}(\mathbf{z}_+, \mathbf{z}_-)$ .

The extended moduli spaces  $\mathcal{M}_{J_+}^{\dagger, ext}(\mathbf{z}_+, \delta_0^r \gamma')$  and  $\mathcal{M}_{J_-}^{\dagger, ext}(\delta_0^r \gamma', \mathbf{z}_-)$  are defined similarly.

*Notation.* A sector of  $D^2$  is *large* if it has angle  $\pi < \phi$  and *small* if it has angle  $0 < \phi < \pi$ . If  $R, R' \subset D^2$  are distinct radial rays and  $I \subset \mathbb{R}$  is an interval, then let

$\mathfrak{S}(R, R'; I)$  be a counterclockwise sector in  $D^2$  from  $R$  to  $R'$  such that the angle of the sector lies in  $I$ ; if we do not specify  $I$ , we write  $\mathfrak{S}(R, R')$ .

*Role of extended moduli spaces.* We briefly explain the role of the extended moduli space  $\mathcal{M}^{\dagger, ext} = \mathcal{M}^{\dagger, ext}(\mathbf{z}_+, \mathbf{z}_-)$ . Let  $\bar{u} \in \mathcal{M}^{\dagger}$ . Then  $(i'_k, j'_k) \rightarrow (i_k, j_k)$  in  $\vec{\mathcal{D}}_+$  corresponds to an end  $\mathcal{E}_{+,i}$  of  $\bar{u}$ , whose projection  $\pi_{D^2}$  to a small neighborhood of  $z_\infty$  in  $\bar{S}$  sweeps out a sector  $\mathfrak{S} = \mathfrak{S}(\bar{u}_{i_k, j_k}, \bar{h}(\bar{u}_{i'_k, j'_k}))$ . If  $\mathfrak{S}$  is large, then  $\pi_{D^2}(\mathcal{E}_{+,i})$  has no slits by the definition of  $\bar{u} \in \mathcal{M}^{\dagger}$  (i.e.,  $\pi_{D^2}$  maps no boundary point of  $\mathcal{E}_{+,i}$  to  $\bar{a}_{i_k, j_k} - \bar{a}_{i_k, j_k}$  or  $\bar{h}(\bar{a}_{i'_k, j'_k} - \bar{a}_{i'_k, j'_k})$ ), and the neighborhood of  $\bar{u}$  in  $\mathcal{M}^{\dagger}$  is generically a codimension  $\geq 1$  submanifold of an extended moduli space  $\mathcal{M}^{\dagger, ext}$ .

**5.8. Transversality.** We first discuss the regularity of almost complex structures on  $\bar{W}$ ,  $\bar{W}'$ ,  $\bar{W}_+$  and  $\bar{W}_-$ .

**5.8.1. Transversality for  $\bar{W}$  and  $\bar{W}'$ .**

**Definition 5.8.1.** The almost complex structure  $J \in \mathcal{J}$  is *regular* if  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}')$  is transversely cut out for all tuples  $\mathbf{y}$  and  $\mathbf{y}'$  in  $\mathbf{a} \cap h(\mathbf{a})$ . The almost complex structure  $\bar{J} \in \bar{\mathcal{J}}$  is *regular* if  $\mathcal{M}_{\bar{J}}^{\dagger, ext}(\mathbf{z}, \mathbf{z}')$  is transversely cut out for all tuples  $\mathbf{z} = \{z_\infty^p(\mathcal{D})\} \cup \mathbf{y}$  and  $\mathbf{z}' = \{z_\infty^q(\mathcal{D}')\} \cup \mathbf{y}'$ .

Note that  $\mathcal{M}_J(\mathbf{y}, \mathbf{y}') = \mathcal{M}_J^s(\mathbf{y}, \mathbf{y}')$  and  $\mathcal{M}_{\bar{J}}^{\dagger, ext}(\mathbf{z}, \mathbf{z}') = \mathcal{M}_{\bar{J}}^{\dagger, ext, s}(\mathbf{z}, \mathbf{z}')$ .

Recall that a generic  $J \in \mathcal{J}$  is regular by Lemma 4.7.2. The same proof also gives:

**Lemma 5.8.2.** *A generic  $\bar{J} \in \bar{\mathcal{J}}$  is regular.*

We write  $\mathcal{J}^{reg} \subset \mathcal{J}$  for the subset of all regular  $J$  and  $\bar{\mathcal{J}}^{reg} \subset \bar{\mathcal{J}}$  for the subset of all regular  $\bar{J}$ .

**Definition 5.8.3.** The almost complex structure  $J' \in \mathcal{J}'$  is *regular* if  $\mathcal{M}_{J'}^s(\gamma, \gamma')$  is transversely cut out (in the Morse-Bott sense) for all  $\gamma$  and  $\gamma'$  in  $\hat{\mathcal{O}}_k$ ,  $k \leq 2g$ . The almost complex structure  $\bar{J}' \in \bar{\mathcal{J}}'$  is *regular* if  $\mathcal{M}_{\bar{J}'}^s(\delta^p \gamma, \delta^q \gamma')$  is transversely cut out for all  $\delta^p \gamma, \delta^q \gamma' \in \bar{\mathcal{O}}_k$ ,  $k \leq 2g$ .

Recall that a generic  $J' \in \mathcal{J}'$  is regular by Lemma 3.5.2. The same proof also gives:

**Lemma 5.8.4.** *A generic  $\bar{J}' \in \bar{\mathcal{J}}'$  is regular.*

We write  $(\mathcal{J}')^{reg} \subset \mathcal{J}'$  for the subset of all regular  $J'$  and  $(\bar{\mathcal{J}}')^{reg} \subset \bar{\mathcal{J}}'$  for the subset of all regular  $\bar{J}'$ .

**5.8.2. Transversality for  $W_+$ ,  $\bar{W}_+$  and  $\bar{W}_-$ .**

**Definition 5.8.5.** The almost complex structure  $J_+ \in \mathcal{J}_+$  is *regular* if the following hold:

- (1) all moduli spaces  $\mathcal{M}_{J_+}(\mathbf{y}, \gamma)$  with  $\mathbf{y}$  a  $k$ -tuple of  $\mathbf{a} \cap h(\mathbf{a})$  and  $\gamma \in \widehat{\mathcal{O}}_k$ ,  $k \leq 2g$ , are transversely cut out (in the Morse-Bott sense in the case of a Morse-Bott building); and
- (2) the restrictions  $J$  and  $J'$  of  $J_+$  to the positive and negative ends belong to  $\mathcal{J}^{reg}$  and  $(\mathcal{J}')^{reg}$ , respectively.

Here every  $u \in \mathcal{M}_{J_+}(\mathbf{y}, \gamma)$  is somewhere injective due to the presence of the HF end. In other words,  $\mathcal{M}_{J_+}(\mathbf{y}, \gamma) = \mathcal{M}_{J_+}^s(\mathbf{y}, \gamma)$ .

**Definition 5.8.6.** The almost complex structure  $\overline{J}_+ \in \overline{\mathcal{J}}_+$  is *regular* if the following hold:

- (1) all moduli spaces  $\mathcal{M}_{\overline{J}_+}^{\dagger, ext}(\mathbf{z}, \delta_0^r \gamma')$  with  $\mathbf{z} = \{z_\infty^p(\mathcal{D})\} \cup \mathbf{y}$ ,  $\gamma' \in \widehat{\mathcal{O}}_*$  and  $\mathbf{y}$  a tuple of  $\mathbf{a} \cap h(\mathbf{a})$ , are transversely cut out; and
- (2) the restrictions  $\overline{J}$  and  $\overline{J}'$  of  $\overline{J}_+$  to the positive and negative ends belong to  $\overline{\mathcal{J}}^{reg}$  and  $(\overline{\mathcal{J}}')^{reg}$ , respectively.

Note that  $\mathcal{M}_{\overline{J}_+}^{\dagger, ext}(\mathbf{z}, \delta_0^r \gamma') = \mathcal{M}_{\overline{J}_+}^{\dagger, ext, s}(\mathbf{z}, \delta_0^r \gamma')$ . The regularity of  $\overline{J}_- \in \overline{\mathcal{J}}_-$  is defined similarly.

We write  $\mathcal{J}_+^{reg}$  for the space of regular  $J_+ \in \mathcal{J}_+$  and  $\overline{\mathcal{J}}_\pm^{reg}$  for the space of regular  $\overline{J}_\pm \in \overline{\mathcal{J}}_\pm$ . Without loss of generality we may assume that the regular  $J_+$  of interest are the restrictions of regular  $\overline{J}_+$ .

*Remark 5.8.7.* The vertical fibers  $\{(s, t)\} \times S$  and  $\{(s, t)\} \times \overline{S}$  are holomorphic, but are not transversely cut out.

**Proposition 5.8.8.** A generic admissible  $J_+$  (resp.  $\overline{J}_\pm$ ) is regular.

*Proof.* We first treat the  $W_+$  case. The proposition follows from a standard transversality argument along the lines of [MS, Theorem 3.1.5], with some modifications. The necessary modifications for almost complex structures  $J' \in \mathcal{J}'$ , defined on  $\mathbb{R} \times N$ , were described in [Hu1, Lemma 9.12(b)], and our situation is almost identical since  $J_+ \subset \mathcal{J}_+$  is the restriction to  $W_+$  of some  $J' \in \mathcal{J}'$ .

The key observation is that each irreducible component of a  $W_+$ -curve  $u : \dot{F} \rightarrow W_+$  is somewhere injective, since each  $[0, 1] \times \{y_i\}$ ,  $y_i \in \mathbf{y}$ , is used exactly once as a positive asymptotic limit. Let  $\pi_N : W_+ \rightarrow N$  be the restriction of the projection  $\pi_N : \mathbb{R} \times N \rightarrow N$  onto the second factor. We then observe that there is a dense open set of points  $p \in \dot{F}$  which are  $\pi_N$ -injective, i.e.,

- (i)  $d(\pi_N \circ u)(p)$  has rank 2; and
- (ii)  $(\pi_N \circ u)(p) = (\pi_N \circ u)(q)$  implies  $p = q$ .

The perturbations to  $J_+$  can then be carried out in a neighborhood of a  $\pi_N$ -injective point  $p \in \dot{F}$  as in [Hu1, Lemma 9.12(b)].

The regularity of the almost complex structures  $J$  and  $J'$  at the ends was already treated, i.e.,  $\mathcal{J}^{reg} \subset \mathcal{J}$  and  $(\mathcal{J}')^{reg} \subset \mathcal{J}'$  are dense by Lemmas 3.5.2 and 4.7.2.

In the  $\overline{W}_+$  case, the perturbations of  $\overline{J}_+$  are allowed on the subset  $U = \overline{W}_+ \cap ((\mathbb{R} \times \overline{N}) - \{\rho \leq \varepsilon\})$  for some small  $\varepsilon > 0$ . We simply observe that all the curves  $\overline{u}$



in the moduli spaces  $\mathcal{M}_{\mathcal{J}_+}^{\dagger, ext}(\mathbf{z}, \delta_0^r \gamma')$  in Definition 5.8.6 pass through  $U$ , and pick a  $\pi_N$ -injective point  $p \in \dot{F}$  such that  $\bar{u}(p) \in U$ . The  $\bar{W}_-$  case is similar.  $\square$

**5.8.3. Some automatic transversality results.** We collect some automatic transversality results.

**Lemma 5.8.9.** *The curve  $\sigma_\infty^- \subset \bar{W}_-$ , viewed as having Lagrangian boundary  $L_{\bar{a}_{i,j}}^-$ , is a regular holomorphic curve with  $\text{ind}(\sigma_\infty^-) = 0$ .*

*Proof.* We first calculate the Fredholm index of the holomorphic embedding  $\bar{u} : \dot{F} \rightarrow \bar{W}_-$  with image  $\sigma_\infty^- \subset \bar{W}_-$ . Here  $\dot{F}$  is a disk with a boundary puncture and an interior puncture. Using the trivialization  $\tau$  from Section 5.7, we compute that  $-\chi(\dot{F}) = 0$ ,  $\mu_\tau(\delta_0, \bar{u}) = 1$ ,  $\mu_\tau(z_\infty) = 1$ ,  $c_1(\bar{u}^* T\bar{S}, \tau) = 0$ . Hence  $\text{ind}(\bar{u}) = 0$  by Equation (5.5.3), which is still valid in the current situation.

We now use the doubling technique from Theorem 5.5.1. The double of  $\dot{F}$  — a sphere with three punctures — is denoted by  $2\dot{F}$  and the double of  $\bar{u}$  is denoted by  $2\bar{u}$ . The index of the doubled operator  $2D_{\bar{u}}$  is  $\text{ind}(2\bar{u}) = 2\text{ind}(\bar{u}) = 0$  and  $D_{\bar{u}}$  is surjective if and only if  $2D_{\bar{u}}$  is surjective.

Now, by Wendl's automatic transversality theorem [We3, Theorem 1],  $2D_{\bar{u}}$  is surjective if

$$(5.8.1) \quad \text{ind}(2\bar{u}) \geq 2g + \#\Gamma_0 - 1,$$

where  $g$  is the genus of  $2\dot{F}$  and  $\#\Gamma_0$  is the count of punctures with even Conley-Zehnder index. In the present situation,  $g = 0$  and  $\#\Gamma_0 = 0$ , so Equation (5.8.1) becomes  $\text{ind}(2\bar{u}) \geq -1$ , which is satisfied.  $\square$

The following is easier, and is stated without proof:

**Lemma 5.8.10.** *If  $\bar{\mathcal{J}}$  is sufficiently close to  $\bar{\mathcal{J}}_0$ , then  $\mathcal{M}_{\bar{\mathcal{J}}}^{\dagger, n^*=1}(\{z_\infty\} \cup \mathbf{y}, \{y_0\} \cup \mathbf{y})/\mathbb{R}$  is transversely cut out and consists of a unique curve which is represented by a thin strip in  $D^2$  from  $z_\infty$  to  $y_0 = x_i$  or  $x'_i$ .*

**5.8.4. Marked points and transversality.** In the definition of the  $\Psi$ -map in Section 7, we consider multisections of  $\bar{W}_-$  which pass through the marked point  $\bar{\mathbf{m}} = ((0, \frac{3}{2}), z_\infty)$ . The marked point  $\bar{\mathbf{m}}$ , however, is *nongeneric*. In order to ensure the regularity of such moduli spaces with respect to  $\bar{\mathbf{m}}$ , we need to enlarge the class of  $\bar{\mathcal{J}}_- \in \bar{\mathcal{J}}_-^{reg}$  to the class of  $\bar{\mathcal{J}}_-^\diamond$ , which we now define.

**Definition 5.8.11.** Let  $\varepsilon > 0$  and let  $U \not\ni \bar{\mathbf{m}}$  be an open set of  $\bar{W}_-$ . Then an almost complex structure  $\bar{\mathcal{J}}_-^\diamond$  on  $\bar{W}_-$  is  $(\varepsilon, U)$ -close to  $\bar{\mathcal{J}}_-$  if:

- $\bar{\mathcal{J}}_-^\diamond = \bar{\mathcal{J}}_-$  on  $\bar{W}_- - U$ ; and
- $\bar{\mathcal{J}}_-^\diamond$  is  $\varepsilon$ -close to  $\bar{\mathcal{J}}_-$  on  $U$ .

Here the  $\varepsilon$ -closeness is measured with respect to a metric  $g$  on  $\bar{W}_-$  which is the restriction of an  $s$ -invariant metric on  $\mathbb{R} \times \bar{N}$ .

**Convention 5.8.12.** Unless stated otherwise:

- $U = \pi_{B_-}^{-1}(B_\delta(p)) - \{\rho \leq \delta\}$  is an open neighborhood of  $K_{p,2\delta} = \pi_{B_-}^{-1}(p) - \{\rho < 2\delta\}$ , where  $\delta > 0$  is arbitrarily small,  $\bar{m}^b \neq p \in B_-$ , and  $B_\delta(p) \subset B_-$  is an open ball of radius  $\delta$  about  $p$ .
- $\bar{J}_-^\diamond$  is  $(\varepsilon, U)$ -close to  $\bar{J}_- \in \bar{J}_-^{reg}$ , where  $\varepsilon > 0$  is arbitrarily small.

Observe that  $U$  is disjoint from the section at infinity. When we want to emphasize  $(\varepsilon, U)$  or  $(\varepsilon, \delta, p)$ , we write  $\bar{J}_-^\diamond(\varepsilon, U)$  or  $\bar{J}_-^\diamond(\varepsilon, \delta, p)$ .

**Definition 5.8.13.** A degree  $k$  almost multisection of  $(\bar{W}_-, \bar{J}_-^\diamond)$  from  $\delta_0^r \gamma$  to  $\mathbf{z}' = \{z_\infty^q(\mathcal{D}')\} \cup \mathbf{y}'$  is a pair  $(\bar{u}, \mathcal{C})$  which is defined in the same way as a degree  $k$  multisection of  $(\bar{W}_-, \bar{J}_-)$ , except that  $\bar{u}$  is a degree  $k$  multisection of

$$\pi_{B_-} : \bar{W}_- - \pi_{B_-}^{-1}(V) \rightarrow B_- - V,$$

where  $V = \pi_{B_-}(U)$ .

Let  $\mathcal{M}_{\bar{J}_-^\diamond}(\delta_0^r \gamma, \mathbf{z}')$  be the moduli space of almost multisections of  $(\bar{W}_-, \bar{J}_-^\diamond)$  from  $\delta_0^r \gamma$  to  $\mathbf{z}'$ . The regularity of  $\bar{J}_-$  and the closeness of  $\bar{J}_-^\diamond$  to  $\bar{J}_-$  imply:

- (i)  $\mathcal{M}_{\bar{J}_-^\diamond}^\dagger(\delta_0^r \gamma, \mathbf{z}')$  is regular;
- (ii)  $\mathcal{M}_{\bar{J}_-^\diamond}^\dagger(\delta_0^r \gamma, \mathbf{z}')$  is close to  $\mathcal{M}_{\bar{J}_-}^\dagger(\delta_0^r \gamma, \mathbf{z}')$ ; and
- (iii) all the boundary strata of  $\mathcal{M}_{\bar{J}_-^\diamond}^\dagger(\delta_0^r \gamma, \mathbf{z}')$  are close to the corresponding boundary strata of  $\mathcal{M}_{\bar{J}_-}^\dagger(\delta_0^r \gamma, \mathbf{z}')$ .

Note that we can still refer to  $n^*(\bar{u})$  since it is a homological quantity.

Let  $K \not\propto \bar{m}$  be a compact set of  $\bar{W}_-$ . We define the modifier  $K$  to mean that  $\bar{u}$  passes through  $K$ .

**Definition 5.8.14.** The almost complex structure  $\bar{J}_-^\diamond$  on  $\bar{W}_-$  is  $K$ -regular with respect to  $\bar{m}$  if all the moduli spaces  $\mathcal{M}_{\bar{J}_-^\diamond}^{\dagger, ext, K}(\delta_0^r \gamma, \mathbf{z}'; \bar{m})$  are transversely cut out.

**Lemma 5.8.15.** A generic  $\bar{J}_-^\diamond$  is  $K_{p,2\delta}$ -regular with respect to  $\bar{m}$ .

*Proof.* The proof is similar to that of [MS, Theorem 3.1.7], with modifications as in Proposition 5.8.8.  $\square$

## 6. THE CHAIN MAP FROM $\widehat{HF}$ TO $PFH$

**6.1. Compactness for  $W_+$ .** In this subsection we treat the compactness of holomorphic curves in  $W_+$  which will be used to establish the chain map  $\Phi$  in Section 6.2.

Suppose  $J_+ \in \mathcal{J}_+$  and  $J, J'$  are the restrictions of  $J_+$  to the positive and negative ends. Let  $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$  and  $\gamma = \prod_{k=1}^l \gamma_k^{m_k} \in \widehat{\mathcal{O}}_{2g}$ .

*In this section, we may pass to a subsequence of a sequence of holomorphic curves without specific mention.*

6.1.1. *Euler characteristic bounds.* We first state a preliminary lemma:

**Lemma 6.1.1.** *Let  $u_i : (\dot{F}_i, j_i) \rightarrow (W_+, J_+)$ ,  $i \in \mathbb{N}$ , be a sequence of  $W_+$ -curves from  $\mathbf{y}$  to  $\gamma$ . Then there is a subsequence such that all the  $\dot{F}_i$  are diffeomorphic to a fixed  $\dot{F}$ .*

*Proof.* The proof is given in two steps.

**Step 1** ( $\omega$ -area bounds). This is a consequence of the vanishing of the flux  $F_h$  of  $h$  (cf. Section 3.3). View the broken closed string  $\gamma_{\mathbf{y}}$  corresponding to  $\mathbf{y}$  as a collection of curves in

$$(L_{\mathbf{a}}^+ \cap \check{W}_+) \cup (\{3\} \times [0, 1] \times \mathbf{y}) \subset \check{W}_+.$$

Then  $\gamma_{\mathbf{y}}$  is uniquely determined up to a homotopy which is supported on  $L_{\mathbf{a}}^+ \cap \check{W}_+$ . Let  $u_i : \dot{F}_i \rightarrow W_+$ ,  $i = 1, 2$ , be two  $W_+$ -curves from  $\mathbf{y}$  to  $\gamma$  and let  $\check{u}_i : \check{F} \rightarrow \check{W}_+$  be their compactifications. Then  $\check{u}_i(\partial_+ \check{F})$  is homotopic to  $\gamma_{\mathbf{y}}$ ,  $i = 1, 2$ , where  $\partial_+ \check{F}$  is the union of boundary components of  $\check{F}$  which map to the positive ( $s > 0$ ) part of  $\check{W}_+$ . Hence  $\check{u}_1 - \check{u}_2$  can be viewed as a closed surface  $Z \in H_2(W_+) \simeq H_2(\check{W}_+) \simeq H_2(N)$ . Since the flux  $F_h$  vanishes, the  $\omega$ -area of a  $W_+$ -curve  $u$  only depends on  $\mathbf{y}$  and  $\gamma$ .

**Step 2** (Genus bounds). Now that we have an  $\omega$ -area bound on the sequence  $\{u_i\}$ , we can apply the Gromov-Taubes compactness theorem [T3, Proposition 3.3], which is a local result and carries over to the symplectic cobordism  $(W_+, \Omega_+)$  without difficulty. As explained in [Hu1, Lemma 9.8], the Gromov-Taubes compactness theorem implies the weak convergence of  $u_i$  as currents to a holomorphic building  $u_{\infty}$ . In particular, we may assume that  $[u_i] \in H_2(W_+, \mathbf{y}, \gamma)$  is fixed for all  $i$ .

We now use the fact that  $[u_i]$  is fixed to bound the genus of  $\dot{F}_i$ . The relative adjunction formula (Lemma 5.6.3) gives:

$$c_1(\check{u}_i^* TW_+, (\tau, \partial_t)) = \chi(\dot{F}_i) - w_{\tau}^-(u_i) + Q_{\tau}(u_i) - 2\delta(u_i).$$

In view of the writhe bound

$$w_{\tau}^-(u_i) \geq \tilde{\mu}_{\tau}(\gamma) - \mu_{\tau}^-(u_i)$$

from [Hu2, Lemma 4.20] and the nonnegativity of  $\delta(u_i)$ , we obtain:

$$(6.1.1) \quad \chi(\dot{F}_i) \geq c_1(\check{u}_i^* TW_+, (\tau, \partial_t)) + \tilde{\mu}_{\tau}(\gamma) - \mu_{\tau}^-(u_i) - Q_{\tau}(u_i).$$

This bounds  $\chi(\dot{F}_i)$  from below, since all the terms on the right-hand side either depend on the homology class of  $u_i$  or the data of the ends. Hence we may assume that all the  $\dot{F}_i$  are diffeomorphic to a fixed  $\dot{F}$ .  $\square$

6.1.2. *SFT compactness.*

**Proposition 6.1.2.** *Let  $u_i : (\dot{F}_i, j_i) \rightarrow (W_+, J_+)$ ,  $i \in \mathbb{N}$ , be a sequence of  $W_+$ -curves from  $\mathbf{y}$  to  $\gamma$ . Then there is a subsequence which converges in the sense of SFT to a level  $a + b + 1$  holomorphic building*

$$u_{\infty} = v_{-b} \cup \cdots \cup v_a,$$

where  $v_j$  is a holomorphic map to  $W_j = W$  for  $j > 0$ ,  $W_0 = W_+$  for  $j = 0$ , and  $W_j = \mathbb{R} \times N$  for  $j < 0$ , and the levels  $W_{-b}, \dots, W_a$  are arranged in order from lowest to highest.

Here “convergence in the SFT sense” means convergence with respect to the topology described in [BEHWZ]. We will also write  $\pi_j : W_j \rightarrow B_j$  for the projection of  $W_j$  to the appropriate base  $B_j = B, B_+$ , or  $B'$ .

*Proof.* By Lemma 6.1.1, we may assume that  $\dot{F}_i = \dot{F}$  as smooth surfaces. We can then apply the SFT compactness theorem from [BEHWZ]. Since we are dealing with holomorphic curves with Lagrangian boundary, we sketch some of the standard details of SFT compactness.

Let  $h$  be a Riemannian metric on  $W_+$  which is compatible with  $J_+$  and is cylindrical at both ends. By [BEHWZ, Lemma 10.7], we can add a bounded number of interior marked points  $Z_i$  to  $\dot{F}$  so that the following holds: Let  $g_i$  be a hyperbolic metric on  $\dot{F} - Z_i$  which is compatible with  $j_i$  and which has geodesic boundary and cusps at the boundary/interior punctures. Then we have a bound on  $\rho \|\nabla u_i\|$ , the so-called “gradient bound”, where the norm  $\|\cdot\|$  and the gradient are measured with respect to the metrics  $g_i$  and  $h$ , and  $\rho$  is the injectivity radius of the surface doubled along  $\partial \dot{F}$ .

The sequence  $u_i$  converges on the thick part by the gradient bound. If  $g_i$  degenerates as  $i \rightarrow \infty$ , then there is a finite collection of homotopy classes of properly embedded arcs and closed curves on  $\dot{F} - Z_i$ , whose geodesic representatives are mutually disjoint, and which are pinched as  $i \rightarrow \infty$ , i.e., the lengths of the geodesic representatives go to zero. Let  $\delta$  be any such homotopy class and let  $\text{Thin}_\varepsilon(\delta, g_i)$  be the  $\varepsilon$ -thin annulus  $\subset (\dot{F} - Z_i, g_i)$  whose core is homotopic to  $\delta$ . Here  $\varepsilon > 0$  is sufficiently small. Then  $u_i(\text{Thin}_\varepsilon(\delta, g_i))$  converges to a “holomorphic sausage” as in [BEHWZ, Figure 14], i.e., a stack of holomorphic cylinders or strips whose ends are either removable or are asymptotic to chords or closed orbits, and whose successive components have ends which are identified. (The same holds for  $u_i$  of cusp pieces of the thin part.)

The limiting curve  $u_\infty$  can be written as a level  $a + b + 1$  holomorphic building  $v_{-b} \cup \dots \cup v_a$ , where each  $v_j$  is not necessarily irreducible and may have nodes. Here (i)  $a, b$  are nonnegative integers, (ii)  $v_j$  is a holomorphic map to  $W_j$ , and (iii) the levels  $v_j$  are ordered from the negative end to the positive end as  $j$  increases. As usual, if the level  $v_j$  is just a union of trivial cylinders, then it will be elided.  $\square$

**6.1.3. Main theorem.** Suppose  $J_+ \in \mathcal{J}_+^{reg}$  and  $J, J'$  are the restrictions of  $J_+$  to the positive and negative ends.

The following is the main theorem of this subsection:

**Theorem 6.1.3.** *Let  $u_i : (\dot{F}_i, j_i) \rightarrow (W_+, J_+)$ ,  $i \in \mathbb{N}$ , be a sequence of  $W_+$ -curves from  $\mathbf{y}$  to  $\gamma$ . If  $I_{W_+}(u_i) = 1$  for all  $i$ , then a subsequence of  $u_i$  converges in the sense of SFT to one of the following:*

- (1) *an  $I_{W_+} = 1$  curve;*
- (2) *a building with two levels consisting of an  $I_{HF} = 1$  curve and an  $I_{W_+} = 0$  curve; or*

- (3) a building with multiple levels consisting of an  $I_{W_+} = 0$  curve, an  $I_{ECH} = 1$  curve, and possible  $I_{ECH} = 0$  connectors in between.

Similarly, if  $I_{W_+}(u_i) = 0$  for all  $i$ , then a subsequence of  $u_i$  converges to an  $I_{W_+} = 0$  curve.

We write “an  $I_{\#} = i$  curve” as shorthand for “a  $\#$ -curve with ECH index  $I_{\#} = i$ ”.

We postpone the proof of the main theorem after a more detailed discussion of the structure of the SFT limit. Let  $u_{\infty}$  be the SFT limit of the sequence  $\{u_i\}$ , given by Proposition 6.1.2. A *ghost component* of  $u_{\infty}$  is an irreducible component of  $u_{\infty}$  which maps to a point. By the SFT compactness theorem, the domain of a ghost component is necessarily a stable Riemann surface. We recall that a Riemann surface  $F$  with  $k_{int}$  interior marked points and  $k_{bdr}$  boundary marked points is stable if:

- $-\chi(F) + k_{int} \geq 1$  when  $\partial F = \emptyset$ , or
- $-2\chi(F) + 2k_{int} + k_{bdr} \geq 1$  if  $\partial F \neq \emptyset$ .

Let us write  $u_{\infty} = v_{-b} \cup \dots \cup v_a \cup u_{\infty}^g$ , where  $v_j$  has no ghost components and  $u_{\infty}^g$  is the union of ghost components. We will also write  $u_{\infty}^{ng} = v_{-b} \cup \dots \cup v_a$ .

*Remark 6.1.4.* The ECH index of a curve depends only on its relative homology class and therefore ghost components do not contribute to it. Hence, by the additivity of ECH indices (Lemma 5.6.8), if  $u_i$  is a sequence of  $J_+$ -holomorphic maps with constant ECH index, then:

$$(6.1.2) \quad \sum_{j=1}^a I_{HF}(v_j) + I_{W_+}(v_0) + \sum_{j=1}^b I_{ECH}(v_{-j}) = I(u_i).$$

**Lemma 6.1.5.** *Each level  $v_j$ ,  $j = -b, \dots, a$ , is a degree  $2g$  multisection of  $\pi_j : W_j \rightarrow B_j$  with no branch points along  $\partial B_j$ .*

*Proof.* Since  $u_i$  is a degree  $2g$  multisection of  $\pi_{B_+} : W_+ \rightarrow B_+$  for all  $i$ , it follows that, with the exception of finitely many  $p \in B_j$ , every level  $v_j$  intersects a fiber  $\pi_j^{-1}(p)$  exactly  $2g$  times.

We show that on any level  $v_j$  there are no irreducible components which lie in a fiber  $\pi_j^{-1}(p)$ . Arguing by contradiction, suppose  $\tilde{v} : \tilde{F} \rightarrow W_j$  is an irreducible component which maps to a fiber  $\pi_j^{-1}(p)$ . If  $p \in \text{int}(B_j)$ , then  $\tilde{v}$  is a holomorphic map from a closed Riemann surface  $\tilde{F}$  to  $\pi_j^{-1}(p)$ . Since  $\pi_j^{-1}(p)$  is a Riemann surface with nonempty boundary,  $\tilde{v}$  must be constant. On the other hand, if  $p \in \partial B_j$ , it is also possible that  $\tilde{F}$  is a compact Riemann surface with nonempty boundary and  $\tilde{v}(\partial \tilde{F}) \subset \mathbf{a}$ . However, since  $S - \mathbf{a}$  is connected and nontrivially intersects  $\partial S$ ,  $\tilde{v}$  must also be constant. Since ghost components are excluded from  $v_j$  by definition, we have a contradiction.

Finally, if  $j \geq 0$ , then we claim that  $\pi_j \circ v_j$  has no branch points along  $\partial B_j$ . This is due to the fact that  $v_0$  uses each component of  $L_{\mathbf{a}}^+$  exactly once and  $v_j$ ,  $j > 0$ , uses each component of  $\mathbb{R} \times \{1\} \times \mathbf{a}$  and each component of  $\mathbb{R} \times \{0\} \times h(\mathbf{a})$  exactly once.  $\square$

**Lemma 6.1.6.** *Let  $u_\infty$  be the SFT limit of a sequence of  $J_+$ -holomorphic multi-sections  $u_i$  with constant ECH index. If  $J_+$  is regular, then:*

- $I_{HF}(v_j) > 0$  for  $j > 0$ ,
- $I_{W_+}(v_0) \geq 0$ , and
- $I_{ECH}(v_j) \geq 0$  for  $j < 0$ .

Moreover, all the  $v_j$ ,  $j \geq 0$ , are somewhere injective and satisfy  $\text{ind}(v_j) \geq 0$ . If  $I_{W_+}(u_i) \leq 1$  in addition, then  $v_j$ ,  $j < 0$ , is somewhere injective and satisfies  $\text{ind}(v_j) \geq 0$ .

*Proof.* Since  $v_j$ ,  $j \geq 0$ , is a degree  $2g$  multisection by Lemma 6.1.5 and uses each connected component of the boundary Lagrangian exactly once, it is somewhere injective. Also since  $J$  and  $J_+$  are regular, it follows that the curves  $v_j$ ,  $j \geq 0$ , are regular. Hence  $\text{ind}_W(v_j) \geq 0$  for  $j > 0$  and  $\text{ind}_{W_+}(v_0) \geq 0$ . Moreover, since  $\text{ind}_W(v_j) = 0$ ,  $j > 0$ , if and only if  $v_j$  is a union of trivial strips, we may assume that  $\text{ind}_W(v_j) > 0$  for all  $j > 0$ . By the index inequality (Theorems 4.5.13 and 5.6.9) we have  $I_{HF}(v_j) > 0$  for  $j > 0$  and  $I_{W_+}(v_0) \geq 0$ . On the other hand, if  $j < 0$ , then  $I_{ECH}(v_j) \geq 0$  by [HT1, Proposition 7.15(a)].

If  $I_{W_+}(u_i) \leq 1$  in addition, then  $I_{ECH}(v_j) \leq 1$  by the additivity of the ECH index. Hence [Hu1, Lemma 9.5] implies that  $v_j$  is somewhere injective and  $\text{ind}(v_j) \geq 0$  since  $J'$  is regular.  $\square$

**Lemma 6.1.7.** *Let  $u_\infty$  be the SFT limit of a sequence of  $J_+$ -holomorphic multi-sections  $u_i$  with  $I_{W_+}(u_i) \leq 1$  for all  $i$ . If  $J_+$  is regular, then  $u_\infty^g = \emptyset$ .*

*Proof.* Let  $\text{ind}(u_\infty^g)$  and  $\text{ind}(u_\infty^{ng})$  be the sum of the Fredholm indices of the irreducible components of  $u_\infty^g$  and  $u_\infty^{ng}$ , respectively. Also let  $F^{ng}$  and  $F^g$  be the domains of  $u_\infty^{ng}$  and  $u_\infty^g$ , respectively. Then  $\text{ind}(u_\infty^g) = -\chi(F^g)$ . If  $u_\infty$  has  $k_{int}$  interior nodes and  $k_{bdr}$  boundary nodes, then

$$(6.1.3) \quad \text{ind}(u_i) = \text{ind}(u_\infty^{ng}) + \text{ind}(u_\infty^g) + 2k_{int} + k_{bdr}$$

$$(6.1.4) \quad = \text{ind}(u_\infty^{ng}) - \chi(F^g) + 2k_{int} + k_{bdr} \leq 1.$$

In fact interior nodes are codimension two phenomena, and boundary nodes are codimension one phenomena. Next, if  $F^g \neq \emptyset$ , then the stability of the Riemann surface implies that

$$(6.1.5) \quad -\chi(F^g) + 2k_1 + k_2 \geq 2.$$

Equations (6.1.4) and (6.1.5) together imply that  $\text{ind}(u_\infty^{ng}) \leq -1$ . Hence we have  $\text{ind}(v_j) \leq -1$  for some  $v_j$ , which is a contradiction of Lemma 6.1.6 for a regular  $J_+$ . The contradiction came from assuming that  $u_\infty^g \neq \emptyset$ .  $\square$

We now finish the proof of Theorem 6.1.3.

*Proof of Theorem 6.1.3.* The proof is based on a classification of the types of allowable buildings  $u_\infty = v_{-b} \cup \dots \cup v_a \cup u_\infty^g$ . We consider the situation of  $I_{W_+}(u_i) = 1$ , leaving the easier  $I_{W_+}(u_i) = 0$  case to the reader.

By Lemmas 6.1.5 and 6.1.7, the limit  $u_\infty$  consists of a building of degree  $2g$  multisections  $v_j$ . Moreover, by Lemma 6.1.6,  $I_{W_j}(v_j) \geq 0$  for all  $j$ . The additivity of ECH indices gives three possibilities for the limit:

- (1)  $u_\infty = v_0$ , where  $v_0$  is a multisection of  $W_+$  and  $I_{W_+}(v_0) = 1$ ;
- (2)  $u_\infty = v_0 \cup v_1$ , where  $v_0$  is a multisection of  $W_+$  with  $I_{W_+}(v_0) = 0$ , and  $v_1$  is a multisection of  $W$  with  $I_{HF}(v_1) = 1$ ;
- (3)  $u_\infty = v_{-b} \cup \dots \cup v_0$ , where  $v_{-b}, \dots, v_{-1}$  are multisections of  $W'$ ,  $v_0$  is a multisection of  $W_+$ , all but one of  $v_{-b}, \dots, v_0$  have  $I = 0$ , and the remaining level has  $I = 1$ .

What is left to prove is that  $I_{ECH}(v_{-b}) = 1$  in Case (3). This follows from considerations of incoming partitions as in [HT1, Proof of Lemma 7.23]. Let  $m_k$  be the multiplicity of the elliptic orbit  $\gamma_k$  in the orbit set  $\gamma$  and let  $\theta_k$  be the rotation of the first return map of  $\gamma_k$ . By [HT1, Definition 1.8], there is partial order  $\geq_{\theta_k}$  on the set of partitions of  $m_k$ , where

$$(a_1, \dots, a_{l_1}) \geq_{\theta_k} (b_1, \dots, b_{l_2})$$

if there is a Fredholm index zero branched cover of  $\mathbb{R} \times \gamma_k$  with positive ends which partition  $m_k$  into  $(a_1, \dots, a_{l_1})$  and negative ends which partition  $m_k$  into  $(b_1, \dots, b_{l_2})$ . By [HT1, Lemma 7.5], the incoming partition  $P_{\gamma_k}^{out}(m_k)$  — the partition which corresponds to  $\gamma_k^{m_k}$  for  $v_{-b}$  — is maximal. Hence  $I_{ECH}(v_{-b}) = 0$  implies that  $v_{-b}$  is a trivial cylinder, which we have already eliminated.  $\square$

**6.2. Definition of  $\Phi$ .** Suppose  $J_+ \in \mathcal{J}_+^{reg}$  and  $J, J'$  are the restrictions of  $J_+$  to the positive and negative ends. In this subsection we define the chain map

$$(6.2.1) \quad \Phi_{J_+} : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}), J) \rightarrow PFC_{2g}(N, J').$$

We will usually suppress  $J, J'$ , and  $J_+$  from the notation.

We first define an approximation of  $\Phi$ :

**Definition 6.2.1.** Let  $\widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))$  be the chain complex generated by  $\mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$ , before quotienting by the equivalence relation  $\sim$  given in Section 4.9.3. We define the map

$$\Phi' : \widehat{CF}'(S, \mathbf{a}, h(\mathbf{a})) \rightarrow PFC_{2g}(N),$$

$$\mathbf{y} \mapsto \sum_{\gamma \in \widehat{\mathcal{O}}_{2g}} \langle \Phi'(\mathbf{y}), \gamma \rangle \cdot \gamma,$$

where  $\langle \Phi'(\mathbf{y}), \gamma \rangle$  is the mod 2 count of  $\mathcal{M}_{J_+}^{I=0}(\mathbf{y}, \gamma)$ . The count is meaningful since  $\mathcal{M}_{J_+}^{I=0}(\mathbf{y}, \gamma)$  is compact by Theorem 6.1.3.

**Proposition 6.2.2.** *The map*

$$\Phi' : \widehat{CF}'(S, \mathbf{a}, h(\mathbf{a})) \rightarrow PFC_{2g}(N),$$

*is a chain map.*

*Proof.* By Theorem 6.1.3 and Lemma 5.4.7,  $\partial\mathcal{M}_{J_+}^{I=1}(\mathbf{y}, \gamma) = A \sqcup B$ , where

$$\begin{aligned} A &= \coprod_{\mathbf{y}' \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}} \left( (\mathcal{M}_J^{I=1}(\mathbf{y}, \mathbf{y}')/\mathbb{R}) \times \mathcal{M}_{J_+}^{I=0}(\mathbf{y}', \gamma) \right), \\ B &= \coprod_{\gamma \in \widehat{\mathcal{O}}_{2g}} \left( \mathcal{M}_{J_+}^{I=0}(\mathbf{y}', \gamma') \times (\mathcal{M}_{J'}^{I=1}(\gamma', \gamma)/\mathbb{R}) \right). \end{aligned}$$

Here we have omitted the potential contributions of connector components for simplicity.

We examine the corresponding gluings of the holomorphic buildings. The first gluing is that of  $(v_1, v_0) \in A$ . This type of gluing was treated by Lipshitz; see Propositions A.1 and A.2 in [Li, Appendix A]. Observe that there are no multiply-covered curves to glue, since each Reeb chord of a  $2g$ -tuple is used exactly once.

The second type of gluing is that of  $(v_0, v_{-b}) \in B$ , with  $I_{ECH} = 0$  connectors  $v_{-1}, \dots, v_{-b+1}$  in between. The curve  $v_0$  is simply-covered since it has an HF end and the curve  $v_{-b}$  is simply-covered since  $I_{ECH} = 1$  (see [HT1, Proposition 7.15]). This type of gluing was treated carefully in [HT1, HT2]. Although the setting there was the gluing for  $\partial^2 = 0$ , in fact most of the work goes towards properly counting  $I_{ECH} = 0$  connectors, i.e., branched covers of trivial cylinders. Their treatment of gluing/counting the  $I_{ECH} = 0$  connectors carries over with little modification to our case. See Section 6.5 for more details.  $\square$

Let  $\delta_x = ([0, 2] \times \{x\})/(2, x) \sim (0, x)$ , where  $x \in \partial S$ ; it is a Reeb orbit in the negative Morse-Bott family  $\mathcal{N}$  which foliates  $\partial N$ . Also let  $(\mathbb{R} \times \delta_x)^+$  be the restriction of  $\mathbb{R} \times \delta_x$  to  $W_+$ .

**Lemma 6.2.3.** *A  $W_+$ -curve  $u$  which has  $x_i$  or  $x'_i$  at the positive end must have  $(\mathbb{R} \times \delta_{x_i})^+$  or  $(\mathbb{R} \times \delta_{x'_i})^+$  as an irreducible component.*

Recall that  $x_i, x'_i$  are components of the Heegaard Floer contact class of  $\xi_{(S, h)}$ .

*Proof.* Let  $v : \dot{F} \rightarrow W_+$  be the irreducible component of  $u$  which has  $x_i$  at the positive end. The component  $v$  may *a priori* have other positive ends besides  $x_i$ . We will show that  $v = (\mathbb{R} \times \delta_{x_i})^+$ . Let  $\pi_S : \{s \geq 3\} \cap W_+ \rightarrow S$  be defined by identifying  $\{s \geq 3\} \cap W_+ = [3, \infty) \times [0, 1] \times S$  and projecting to the third factor. The composition  $\pi_S \circ v$  is holomorphic.

Let  $(r, \theta)$  be polar coordinates on a small neighborhood  $N(x_i) \subset S$  of  $x_i$  so that  $h(a_i) = \{\theta = -\frac{\pi}{4}\}$ ,  $a_i = \{\theta = 0\}$ , and  $N(x_i) = \{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, r > 0\}$ . If  $v$  is not the restriction of a trivial cylinder, then  $\pi_S \circ v$  must map a neighborhood of the puncture of  $\dot{F}$  corresponding to  $x_i$  to a sector  $\{0 \leq \theta \leq k\pi - \frac{\pi}{4}, r > 0\}$  or  $\{\pi \leq \theta \leq (k+1)\pi - \frac{\pi}{4}, r > 0\}$ , where  $k \geq 1$ . Since such a sector cannot be contained in  $S$ , the map  $\pi_S \circ v|_{s \geq C}$  must map identically to  $x_i$ , where  $C \gg 0$ . By the unique continuation property,  $v = (\mathbb{R} \times \delta_{x_i})^+$ .  $\square$

**Theorem 6.2.4.** *The map  $\Phi'$  descends to a chain map*

$$\Phi : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})) \rightarrow PFC_{2g}(N)$$



which maps the Heegaard Floer contact invariant for  $\xi_{(S,h)}$  to the ECH contact invariant for  $\xi_{(S,h)}$ .

*Proof.* Let  $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ . Assume without loss of generality that  $y_1 = x_1$ . Then  $\mathbf{y}$  is equivalent to  $\mathbf{y}' = \{x'_1, y_2, \dots, y_{2g}\}$ . Let  $u$  be an  $I_{W_+} = 0$  Morse-Bott building from  $\mathbf{y}$  to a generator  $\gamma$  of  $PF C_{2g}(N)$ . Then  $(\mathbb{R} \times \delta_{x_1})^+$  is an irreducible component of  $u$  by Lemma 6.2.3 and  $\gamma$  can be written as  $e\gamma'$ , where  $e$  is the elliptic orbit of the negative Morse-Bott family  $\mathcal{N}$ . Now, by replacing  $(\mathbb{R} \times \delta_{x_1})^+$  by  $(\mathbb{R} \times \delta_{x'_1})^+$  and the augmenting gradient trajectory from  $\delta_{x_1}$  to  $e$  by the augmenting trajectory from  $\delta_{x'_1}$  to  $e$ , we obtain an  $I_{W_+} = 0$  Morse-Bott building from  $\mathbf{y}'$  to  $\gamma = e\gamma'$ . Hence there is a one-to-one correspondence between  $W_+$ -curves from  $\mathbf{y}$  to  $\gamma$  and  $W_+$ -curves from  $\mathbf{y}'$  to  $\gamma$ . Since  $\Phi'(\mathbf{y}) = \Phi'(\mathbf{y}')$ , the map  $\Phi'$  descends to  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ .

The Heegaard Floer contact invariant is given by the equivalence class of  $\mathbf{x} = \{x_1, \dots, x_{2g}\}$  and the ECH contact invariant is given by  $e^{2g}$ . By Lemma 6.2.3, the only  $I_{W_+} = 0$  Morse-Bott building from  $\mathbf{x}$  consists of  $(\mathbb{R} \times \delta_{x_i})^+, i = 1, \dots, 2g$ , augmented by connecting gradient trajectories from  $\delta_{x_i}$  to  $e$  in the Morse-Bott family  $\mathcal{N}$ . Hence  $\Phi'(\mathbf{x}) = e^{2g}$ . Then  $\Phi$  maps the equivalence class of  $\mathbf{x}$  to  $e^{2g}$ .  $\square$

**6.3. Spin<sup>c</sup>-structures.** Let  $\mathcal{S}_{\mathbf{a},h(\mathbf{a})}$  be the set of  $2g$ -tuples of  $\mathbf{a} \cap h(\mathbf{a})$  and let  $\overline{\mathcal{S}}_{\mathbf{a},h(\mathbf{a})} = \mathcal{S}_{\mathbf{a},h(\mathbf{a})} / \sim$ , where  $\{x_i\} \cup \mathbf{y}' \sim \{x'_i\} \cup \mathbf{y}'$  for all  $(2g-1)$ -tuples  $\mathbf{y}'$ .

We define a map

$$\mathfrak{h}'_+ : \mathcal{S}_{\mathbf{a},h(\mathbf{a})} \rightarrow H_1(W_+, \partial_h W_+)$$

as follows: Given  $\mathbf{y} = \{y_1, \dots, y_{2g}\} \in \mathcal{S}_{\mathbf{a},h(\mathbf{a})}$ , where  $y_i \in a_i \cap h(a_{\sigma(i)})$  for some permutation  $\sigma_{2g}$ , we define  $\mathfrak{h}'_+(\mathbf{y})$  as the homology class of the broken closed string obtained by concatenating the following oriented arcs:

- for each  $i \in \{1, \dots, 2g\}$ , the arc  $\{3\} \times [0, 1] \times \{y_i\}$ , with orientation given by  $\partial_t$ ;
- for each  $i \in \{1, \dots, 2g\}$ , an arc from  $\{3\} \times \{1\} \times \{y_i\}$  to  $\{3\} \times \{0\} \times \{y_{\sigma^{-1}(i)}\}$  contained in  $L_{a_i}^+$ .

The homology class  $\mathfrak{h}'_+(\mathbf{y})$  is well-defined since the Lagrangians  $L_{a_i}^+$  are simply-connected.

The map  $\mathfrak{h}'_+$  descends to a map

$$\mathfrak{h}_+ : \overline{\mathcal{S}}_{\mathbf{a},h(\mathbf{a})} \rightarrow H_1(W_+, \partial_h W_+),$$

since the components  $x_i$  and  $x'_i$  of the contact class are both converted into broken closed strings which are nullhomologous in  $H_1(W_+, \partial_h W_+)$ .

The following lemma is an immediate consequence of the definition of  $W_+$ -curves.

**Lemma 6.3.1.** *Let  $\gamma$  be an orbit set at the negative end of  $W_+$  with total homology class  $[\gamma] \in H_1(W_+, \partial_h W_+)$ . Then  $\mathcal{M}_{J_+}(\mathbf{y}, \gamma) \neq \emptyset$  implies that  $\mathfrak{h}_+(\mathbf{y}) = [\gamma]$ .*

There are natural isomorphisms

$$H_1(W_+, \partial_h W_+) \cong H_1(N, \partial N) \cong H_1(M).$$

With respect to these isomorphisms, the total homology class in  $H_1(W_+, \partial_h W_+)$  of an orbit set  $\gamma$  at the negative end of  $W_+$  corresponds to the usual total homology class of  $\gamma$  in  $H_1(M)$ . Moreover,  $\mathfrak{h}_+(\mathbf{y}) = \mathfrak{h}(\mathbf{y})$ , where  $\mathfrak{h}$  is as defined in Section 4.10.

Combining Proposition 4.10.1 and Lemma 6.3.1, we obtain the following theorem:

**Theorem 6.3.2.** *The chain map  $\Phi$  respects the splitting according to  $\text{Spin}^c$ -structures, i.e.,  $\Phi$  is the direct sum of maps*

$$\Phi_A : \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}_\xi + PD(A)) \rightarrow PFC_{2g}(N, A).$$

**6.4. Twisted coefficients.** Let  $\underline{PFH}_{2g}(N, A)$  be the twisted coefficient version of  $PFH_{2g}(N, A)$ , defined as in Section 2.4. For any homology class  $A \in H_1(M)$  we can define a map

$$\underline{\Phi}_A : \widehat{\underline{CF}}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}_\xi + PD(A)) \rightarrow \underline{PFC}_{2g}(N, A)$$

given in the following paragraphs. Choose a  $2g$ -tuple of intersection points  $\mathbf{y}_0$  such that  $s_z(\mathbf{y}_0) = \mathfrak{s}_\xi + PD(A)$  and a complete set of paths  $\{C_{\mathbf{y}}\}$  for  $\mathfrak{s}_\xi + PD(A)$  based at  $\mathbf{y}_0$ .

Let  $\pi_N : W_+ \rightarrow N$  be the restriction of the projection  $\mathbb{R} \times N \rightarrow N$ ,  $(s, x) \mapsto x$ , and let  $\Gamma \subset N$  be the projection by  $\pi_N$  of a broken closed string associated to  $\mathbf{y}_0$ . By Lemma 6.3.1,  $[\Gamma] = A$ . We choose a complete set of paths  $\{C_\gamma\}$  for  $A$  based at  $\Gamma$ .

The projection  $\pi_N$  associates a well-defined homology class in  $H_2(N) \cong H_2(M)$  to any 2-chain in  $\tilde{W}_+$  whose boundary consists of a broken closed string corresponding to  $\mathbf{y}_0$  on one side and  $\Gamma$  on the other side. Then we can define maps

$$\mathfrak{A}_+ : H_2(W_+, \mathbf{y}, \gamma) \rightarrow H_2(M)$$

for all  $\mathbf{y}$  such that  $s_z(\mathbf{y}_0) = \mathfrak{s}_\xi + PD(A)$  and  $\gamma$  such that  $[\gamma] = A$  by  $\mathfrak{A}_+(C) = (\pi_N)_*[C_\gamma \cup C \cup -C_{\mathbf{y}}]$ .

**Definition 6.4.1.** We define the  $\mathbb{F}[H_2(M; \mathbb{Z})]$ -linear map

$$\underline{\Phi}_A : \widehat{\underline{CF}}(S, \mathbf{a}, h(\mathbf{a}), \mathfrak{s}_\xi + PD(A)) \rightarrow \underline{PFC}_{2g}(N, A)$$

by

$$\underline{\Phi}_A(\mathbf{y}) = \sum_{\gamma \in \widehat{\mathcal{O}}_{2g}} \sum_{C \in H_2(W_+, \mathbf{y}, \gamma)} \# \mathcal{M}_{J_+}^{I=0}(\mathbf{y}, \gamma, C) e^{\mathfrak{A}_+(C)} \gamma.$$

**Theorem 6.4.2.** *The map  $\underline{\Phi}_A$  is a chain map.*

*Proof.* The proof is the same as that of Theorem 6.2.4, plus some bookkeeping of the homology classes of the holomorphic maps involved.  $\square$

**6.5. Gluing.** We explain here how to glue the pair  $(v_0, v_{-1})$  consisting of a  $W_+$ -curve  $v_0$  with  $I(v_0) = 0$  and an ECH curve  $v_{-1}$  with  $I(v_{-1}) = 1$ , by inserting branched covers of trivial cylinders. The procedure of gluing two  $I_{ECH} = 1$  curves, as explained in Hutchings-Taubes [HT1, HT2], applies with very little modification.

6.5.1. *Close to breaking.* We first make precise what we mean by a holomorphic curve which is “close to breaking”. We treat  $W'$ , leaving the analogous definitions for  $W$ ,  $W_+$ , and  $W_-$  to the reader.

Choose an  $s$ -invariant Riemannian metric  $g$  on  $W'$ .

**Definition 6.5.1.** Given  $\kappa, \nu > 0$ , two curves  $u_i : (F_i, j_i) \rightarrow W'$ ,  $i = 1, 2$ , are  $(\kappa, \nu)$ -close if there exists a  $(1 + \nu)$ -quasiconformal homeomorphism  $\phi : F_1 \xrightarrow{\sim} F_2$  with respect to  $(j_1, j_2)$  such that

$$\sup_{x \in F_1} |u_1(x) - u_2 \circ \phi(x)| \leq \kappa,$$

where the distance  $|\cdot|$  is measured with respect to  $g$ .

The following is similar to [HT1, Definition 1.10], although the phrasing is slightly different.

**Definition 6.5.2.** Let  $\kappa > 0$  and  $\nu \geq 0$ . A curve  $u$  in  $W'$  is  $(\kappa, \nu)$ -close to an  $(a+b+1)$ -level building  $u_\infty = v_{-b} \cup \dots \cup v_a$  if there exist  $R^{(-2b+1)}, \dots, R^{(2a)} > \frac{1}{\kappa}$  such that, after a suitable translation of  $u$  in the  $s$ -direction which we still call  $u$ , each of the pairs below is  $(\kappa, \nu)$ -close:

- $u|_{-R^{(0)} \leq s \leq R^{(1)}}$  and  $v_0|_{-R^{(0)} \leq s \leq R^{(1)}}$ ;
- $u|_{R^{(1)} \leq s \leq R^{(1)} + R^{(2)}}$  and the restriction of a collection of branched covers of trivial cylinders to  $R^{(1)} \leq s \leq R^{(1)} + R^{(2)}$ ;
- $u|_{R^{(1)} + R^{(2)} \leq s \leq R^{(1)} + 2R^{(2)} + R^{(3)}}$  and an  $s \mapsto s + R^{(1)} + 2R^{(2)}$  translate of  $v_1|_{-R^{(2)} \leq s \leq R^{(3)}}$ ;
- $u|_{R^{(1)} + 2R^{(2)} + R^{(3)} \leq s \leq R^{(1)} + 2R^{(2)} + R^{(3)} + R^{(4)}}$  and the restriction of a collection of branched covers of trivial cylinders to  $R^{(1)} + 2R^{(2)} + R^{(3)} \leq s \leq R^{(1)} + 2R^{(2)} + R^{(3)} + R^{(4)}$ ;

and so on.

Note that, in the case of  $u$  in  $W_+$  or  $W_-$ , we do not need to (and indeed we cannot) translate  $u$  in the  $s$ -direction.

6.5.2. *Review of [HT1, HT2].* We now summarize the Hutchings-Taubes proof of  $\partial^2 = 0$  and discuss the small changes that need to be carried out.

Let  $(Y, \xi = \ker \lambda)$  be a closed 3-manifold with a nondegenerate contact form  $\lambda$  and corresponding Reeb vector field  $R = R_\lambda$ . Let  $(\mathbb{R} \times Y, d(e^s \lambda))$  be the symplectization of  $Y$ , where  $s$  denotes the  $\mathbb{R}$ -coordinate, and let  $J$  be an adapted almost complex structure on  $\mathbb{R} \times Y$ .

Let  $\alpha = (\alpha_1, \dots, \alpha_k)$  and  $\beta = (\beta_1, \dots, \beta_l)$  be ordered sets of Reeb orbits of  $R$ . Here the Reeb orbits may not be embedded and may also be repeated. Then let  $\mathcal{M}_J(\alpha, \beta)$  be the moduli space of finite energy  $J$ -holomorphic curves in  $(\mathbb{R} \times Y, J)$  from  $\alpha$  to  $\beta$ . Let  $\{\gamma_1, \gamma_2, \dots\}$  be the list of simple orbits of  $R$ . For each simple orbit  $\gamma_i$ , we tally the total multiplicity  $m_i(\alpha)$  of  $\gamma_i$  in  $\alpha$ . In this way we can assign an orbit set  $\gamma(\alpha) = \prod_i \gamma_i^{m_i(\alpha)}$  to  $\alpha$ .

We want to glue the pair  $(u_+, u_-)$ , where  $u_+ \in \mathcal{M}_J^{I=1}(\alpha_+, \beta_+)$  and  $u_- \in \mathcal{M}_J^{I=1}(\beta_-, \alpha_-)$ , provided  $\gamma(\beta_+) = \gamma(\beta_-)$ . Since  $I(u_+) = I(u_-) = 1$ , the

curves  $u_+$  and  $u_-$  satisfy the *partition condition* at  $\beta_+$  and  $\beta_-$  for a generic  $J$  (cf. Definition 7.11 and Proposition 7.14 of [HT1]). In particular, for each simple orbit  $\gamma_i$ , the total multiplicity  $m_i(\beta_+) = m_i(\beta_-)$  completely determines the number of ends of  $u_+$  or  $u_-$  going to a cover of  $\gamma_i$ , together with their individual multiplicities. For  $u_+$  (resp.  $u_-$ ), this is encoded by the *incoming partition* (resp. *outgoing partition*) and is denoted by  $P_{\gamma_i}^{\text{in}}(m_i(\beta_+))$  (resp.  $P_{\gamma_i}^{\text{out}}(m_i(\beta_-))$ ). If  $P_{\gamma_i}^{\text{in}}(m_i(\beta_+)) = (a_1, \dots, a_r)$ , then  $u_+$  has  $r$  ends which go to a cover of  $\gamma_i$  with covering multiplicities  $a_1, \dots, a_r$ .

Since  $P_{\gamma_i}^{\text{in}}(m_i(\beta_+)) \neq P_{\gamma_i}^{\text{out}}(m_i(\beta_-))$  in general, we need to insert branched covers of cylinders  $\mathbb{R} \times \gamma_i$  in order to be able to glue  $u_+$  to  $u_-$ . Such a branched cover  $\pi : \Sigma \rightarrow \mathbb{R} \times S^1$  must have  $\beta_+$  at the positive end and  $\beta_-$  at the negative end. A Fredholm index count ([HT1, Lemma 1.7]) implies that  $\Sigma$  must have genus zero; moreover, the partition condition is equivalent to saying that the Fredholm index of the composition  $(\text{id}, \gamma_i) \circ \pi : \Sigma \rightarrow \mathbb{R} \times Y$  is zero, where  $(\text{id}, \gamma_i) : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times Y$  is the holomorphic map for the trivial cylinder  $\mathbb{R} \times \gamma_i$ .

We now give the steps of the Hutchings-Taubes gluing construction. For simplicity, we assume that  $\gamma(\beta_+) = \gamma(\beta_-) = \gamma_1^{m_1}$  and  $\gamma_1$  is elliptic.

*Step 1: Form a preglued curve* (cf. [HT2, Section 5.2]). This is done as follows:

- (1) Choose gluing constants  $0 < h < 1$  and  $r \gg 1/h$ .
- (2) Let  $\mathcal{M}$  be the moduli space of connected, genus zero branched covers  $\pi : \Sigma \rightarrow \mathbb{R} \times S^1$  which have positive and negative ends determined by  $P_{\gamma_1}^{\text{in}}(m_1)$  and  $P_{\gamma_1}^{\text{out}}(m_1)$ . With  $r, h$  fixed, choose the “gluing parameters” which consist of  $\pi \in \mathcal{M}$  and real numbers  $T_+, T_- \geq 5r$ .
- (3) Given  $\pi : \Sigma \rightarrow \mathbb{R} \times S^1$ , let  $\mathfrak{b} \subset \mathbb{R} \times S^1$  be the set of branch points of  $\pi$ . Then let  $s_+ = \max_{b \in \mathfrak{b}} s(b) + 1$  and  $s_- = \min_{b \in \mathfrak{b}} s(b) - 1$ . Let  $\Sigma'$  be the restriction of  $\Sigma$  to  $s_- - T_- \leq s \leq s_+ + T_+$ . The components of  $\Sigma' \cap \{s_+ \leq s \leq s_+ + T_+\}$  are cylinders of “height”  $T_+$ .
- (4) Fix  $\kappa > 0$  sufficiently small. We choose a representative  $u_+$  in  $[u_+] \in \mathcal{M}_{J=1}^{I=1}(\alpha_+, \beta_+)/\mathbb{R}$  such that each component of  $u_+|_{s \leq 0}$  is  $(\kappa, 0)$ -close to a cylinder over some multiple cover of  $\gamma_1$ ; here the multiplicities are given by  $P_{\gamma_1}^{\text{in}}(m_1)$ . Similarly, choose  $u_-$  so that each component of  $u_-|_{s \geq 0}$  is  $(\kappa, 0)$ -close to a cylinder over some multiple cover of  $\gamma_1$ .
- (5) Let  $u_{+T}$  be the  $(s_+ + T_+)$ -translate of  $u_+$  in the  $\mathbb{R}$ -direction and let  $u'_{+T}$  be the restriction of  $u_{+T}$  to  $s \geq s_+ + T_+$ . Similarly, let  $u_{-T}$  be the  $(s_- - T_-)$ -translate of  $u_-$  in the  $\mathbb{R}$ -direction and let  $u'_{-T}$  be the restriction of  $u_{-T}$  to  $s \leq s_- - T_-$ .
- (6) The domain  $C_*$  of the preglued curve is  $C'_{+T} \cup \Sigma' \cup C'_{-T}$ , where  $C'_{\pm T}$  are domains for  $u'_{\pm T}$ , modulo the identifications along their boundary components. (To do this correctly, we need asymptotic markers at the ends.) The preglued map  $u_*$  is defined explicitly (see [HT2, Equations (5.5) and (5.6)]) via a cutoff function

which allows us to interpolate between  $u_{\pm T}$  and  $\Sigma$  in the regions  $s_+ \leq s \leq s_+ + T_+$  and  $s_- - T_- \leq s \leq s_-$ . This cutoff function involves the constants  $h$  and  $r$ .

*Step 2: Deform the preglued curve* (cf. [HT2, Section 5.3]). We choose “exponential maps”  $e_-, e, e_+$  which are obtained by flowing in the directions “normal” to  $u_-, u_\Sigma, u_+$  in  $\mathbb{R} \times Y$ . (Here  $u_\Sigma$  is the composition of  $\pi$  with  $\mathbb{R} \times S^1 \rightarrow \mathbb{R} \times Y$  as a trivial cylinder.) The exponential maps can be glued to give the exponential map  $e_*$  which maps to a small tubular neighborhood of  $u_*$  in  $\mathbb{R} \times Y$ . Also choose the cutoff function  $\tilde{\beta}_+ : C_* \rightarrow [0, 1]$ <sup>10</sup> which equals 1 on  $C'_{+T}$ , interpolates between 0 and 1 on the cylinders  $s_+ \leq s \leq s_+ + T_+$  in  $\Sigma'$  and equals 0 elsewhere. Similarly define  $\tilde{\beta}_\Sigma$  and  $\tilde{\beta}_-$ .

Let  $(\psi_+, \psi_\Sigma, \psi_-)$  be triple, where  $\psi_\pm$  is a section of the normal bundle of  $u_{\pm T}$  and  $\psi_\Sigma$  is a function  $\Sigma \rightarrow \mathbb{C}$ . The deformation of  $u_*$  with respect to  $(\psi_+, \psi_\Sigma, \psi_-)$  is a map  $C_* \rightarrow \mathbb{R} \times Y$  given by

$$x \mapsto e_*(x, \tilde{\beta}_- \psi_- + \tilde{\beta}_\Sigma \psi_\Sigma + \tilde{\beta}_+ \psi_+).$$

*Step 3:* We now consider the equation for the deformation to be  $J$ -holomorphic. This equation has the form:

$$(6.5.1) \quad \tilde{\beta}_- \Theta_-(\psi_-, \psi_\Sigma) + \tilde{\beta}_\Sigma \Theta_\Sigma(\psi_-, \psi_\Sigma, \psi_+) + \tilde{\beta}_+ \Theta_+(\psi_\Sigma, \psi_+) = 0.$$

See Equations (5.11), (5.12) and (5.13) of [HT2] for explicit expressions of  $\Theta_-$ ,  $\Theta_+$  and  $\Theta_\Sigma$ .

The strategy is to solve three equations separately:

$$(6.5.2) \quad \Theta_-(\psi_-, \psi_\Sigma) = 0 \quad \text{on all of } u_{-T};$$

$$(6.5.3) \quad \Theta_+(\psi_\Sigma, \psi_+) = 0 \quad \text{on all of } u_{+T};$$

$$(6.5.4) \quad \Theta_\Sigma(\psi_-, \psi_\Sigma, \psi_+) = 0 \quad \text{on all of } \Sigma.$$

In [HT2, Proposition 5.6] it is shown that for sufficiently small  $\psi_\Sigma$  (in a suitable Banach space) there exist maps  $\psi_\pm$  such that  $\psi_\pm = \psi_\pm(\psi_\Sigma)$  solves Equations (6.5.2) and (6.5.3).

We can then view Equation (6.5.4) as an equation in the variable  $\psi_\Sigma$  on all of  $\Sigma$ :

$$(6.5.5) \quad \Theta_\Sigma(\psi_-(\psi_\Sigma), \psi_\Sigma, \psi_+(\psi_\Sigma)) = 0.$$

We then decompose Equation (6.5.5) into two equations (cf. Equations (5.37) and (5.38) in [HT2]):

$$(6.5.6) \quad D_\Sigma \psi_\Sigma + (1 - \Pi) \mathcal{F}_\Sigma(\psi_\Sigma) = 0,$$

$$(6.5.7) \quad \Pi \mathcal{F}_\Sigma(\psi_\Sigma) = 0$$

See [HT2, Section 5.7] for the description of the terms. Equation (6.5.6) has a unique solution by [HT2, Proposition 5.7]. Hence the problem reduces to solving Equation (6.5.7).

<sup>10</sup>The notation in [HT2] is  $\beta_+$ . Here write  $\tilde{\beta}_+$  to distinguish it from the sets of orbits in Section 6.5.2.

This is shown to be equivalent to finding the zeros of a section of the associated obstruction bundle, defined in [HT1, Section 2]. Briefly, the obstruction bundle  $\mathcal{O} \rightarrow [5r, \infty)^2 \times \mathcal{M}$  has fiber

$$\mathcal{O}_{(T_+, T_-, \Sigma)} = \text{Hom}(\text{Coker}(D_\Sigma), \mathbb{R}),$$

where  $D_\Sigma$  is a linearized  $\bar{\partial}$ -type operator on  $\Sigma$  (see [HT1, Section 2.3]), and the section  $\mathfrak{s} : [5r, \infty)^2 \times \mathcal{M} \rightarrow \mathcal{O}$  is given by:

$$\mathfrak{s}(T_+, T_-, \Sigma)(\sigma) = \langle \sigma, \mathcal{F}_\Sigma(\psi_\Sigma) \rangle,$$

where  $\sigma \in \text{Coker}(D_\Sigma)$  and  $\psi_\Sigma$  is the solution to Equation (6.5.6) corresponding to the gluing parameters  $(T_+, T_-, \Sigma)$ .

*Step 4:* Let  $\mathcal{G}_{\kappa, \nu}(u_+, u_-)$  be the set of  $J$ -holomorphic maps which are  $(\kappa, \nu)$ -close to breaking into  $(u_+, u_-)$  (see [HT2, Section 7.1]) and let  $\mathcal{U}_{\kappa, \nu} \subset [5r, \infty)^2 \times \mathcal{M}$  be the set of  $(T_+, T_-, \Sigma)$  such that the corresponding pre-glued curve  $u_*(T_+, T_-, \Sigma)$  (as given in Step 1) is  $(\kappa, \nu)$ -close to breaking.

It remains to show that  $\mathcal{G}_{\kappa, \nu}(u_+, u_-)$  is homeomorphic to  $\mathfrak{s}^{-1}(0) \cap \mathcal{U}_{\kappa, \nu}$  for  $r > 0$  sufficiently large and  $\kappa, \nu$  sufficiently small. This is the content of [HT2, Theorem 7.3].

*Step 5:* We then deform the section  $\mathfrak{s}$  to a linearized section  $\mathfrak{s}_0$  with the same count of zeros. Here  $\mathfrak{s}_0$  only depends on  $\Sigma$  and the linearized  $\bar{\partial}$ -operator  $D_\Sigma$  (which in turn depends on a neighborhood of  $\gamma_1$ ), but not on the actual choices of  $u_+$  and  $u_-$ . The key point to check is that, during the deformation  $(\mathfrak{s}_t)_{t \in [0, 1]}$  from  $\mathfrak{s}_1 = \mathfrak{s}$  to  $\mathfrak{s}_0$ , the zeros of  $\mathfrak{s}_t$  do not cross the boundary  $\{T_+ = 5r\} \cup \{T_- = 5r\}$  of the moduli space  $[5r, \infty)^2 \times \mathcal{M}$ . This is guaranteed by [HT2, Proposition 8.2]. The combinatorial formula for the algebraic count of zeros is given by [HT1, Theorem 1.13].

**6.5.3. The  $\Phi$ -map.** We now turn to gluing the pair  $(v_0, v_{-1})$ , where  $v_0$  is a  $W_+$ -curve with  $I(v_0) = 0$  and  $v_{-1}$  is an ECH curve with  $I(v_{-1}) = 1$ . Suppose the negative end of  $v_0$  is given by  $\beta_+$ , the positive end of  $v_{-1}$  is given by  $\beta_-$ , and  $\gamma(\beta_+) = \gamma(\beta_-) = \gamma_1^{m_1}$ . In our case, there are a few things to check:

(1) The curves  $v_0$  and  $v_{-1}$  must satisfy the partition conditions at their negative and positive ends, respectively. This is a consequence of the ECH index inequality, i.e., Theorem 5.6.9.

(2) Since the  $W_+$ -curve  $v_0$  is not  $s$ -translation invariant, we pick  $s_0$  so that each component of  $v_0|_{s \leq s_0}$  is  $(\kappa, 0)$ -close to a cylinder over some multiple cover of  $\gamma_1$ . (We may still assume that  $v_{-1}$  satisfies the condition that  $v_{-1}|_{s \geq 0}$  is  $(\kappa, 0)$ -close to a cylinder.) Given the gluing parameters  $(T_+, T_-, \Sigma)$ , we take

- $v_0|_{s \geq s_0}$ ;
- $\Sigma'$  shifted by  $s_0 - (s_+ + T_+)$ ; and
- $v_{-1}|_{s \leq 0}$  shifted by  $(s_- - T_-) + s_0 - (s_+ + T_+)$ ;

and preglue.

(3) [HT2, Proposition 5.6] allows us to solve Equations (6.5.2) and (6.5.3) in terms of  $\psi_\Sigma$ . The inputs for [HT2, Proposition 5.6] are [HT2, Lemmas 5.3 and 5.4], which are consequences of the fact that the linearized  $\bar{\partial}$ -operators corresponding to  $v_0$  and  $v_{-1}$  are Fredholm and surjective; this also holds in our case with Lagrangian boundary conditions. Hence Step 3 extends easily to our setting.

The remaining steps carry over with very little change.

**6.6. The variant  $\tilde{\Phi}$ .** In this subsection we define a complex  $\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}))$  which is closely related to  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$  and a variant

$$\tilde{\Phi} : \widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})) \rightarrow PFC_{2g}(N)$$

of  $\Phi$ . This will be useful in the proof of Theorem II.3.3.1.

**6.6.1. The chain complex  $\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}))$ .** Let  $\bar{J} \in \bar{\mathcal{J}}^{reg}$ . We recall some notation introduced in Subsection 4.9.3: Let  $\mathcal{I} \subset \{1, \dots, 2g\}$  be a subset,  $\mathcal{I}^c$  its complement, and  $\mathfrak{S}_{\mathcal{I}^c}$  the group of permutations of  $\mathcal{I}^c$ .

**Definition 6.6.1.** We define the chain complex  $(\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}), \bar{J}), \tilde{\partial})$  generated by the  $2g$ -tuples  $\{z_{\infty, i}\}_{i \in \mathcal{I}} \cup \mathbf{y}'$ , where  $\mathcal{I} \subset \{1, \dots, 2g\}$ ,  $\mathbf{y}' = \{y'_i\}_{i \in \mathcal{I}^c}$ , and the following hold:

- $z_{\infty, i}$  is viewed as an intersection point of  $\bar{a}_i$  and  $\bar{h}(\bar{a}_i)$  and
- $y'_i \in a_i \cap h(a_{\sigma(i)})$  for some  $\sigma \in \mathfrak{S}_{\mathcal{I}^c}$ .

The differential  $\tilde{\partial}$  counts  $I = 1$  multisections  $\bar{u}$  in  $\bar{W}$  with  $n(\bar{u}) \leq 1$ , which satisfy one extra condition, i.e., if we write  $\bar{u} = \bar{u}' \cup \bar{u}''$  (according to the notation introduced in Section 5.7.2), then  $\bar{u}'$  has empty branch locus. The homology of  $(\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a}), \bar{J}), \tilde{\partial})$  is denoted by  $\widetilde{HF}(S, \mathbf{a}, h(\mathbf{a}))$ .

In the differential  $\tilde{\partial}$ , with the exception of trivial strips, we are counting the following curves:

- (1) thin strips from  $z_{\infty, i}$  to either  $x_i$  or  $x'_i$ ; and
- (2)  $I = 1$  curves whose projections to  $\bar{S}$  have image in  $S$ .

**Proposition 6.6.2.** *There is an isomorphism of chain complexes:*

$$\kappa : (\widehat{CF}(\Sigma, \beta, \alpha, z), \hat{\partial}) \xrightarrow{\sim} (\widetilde{CF}(S, \mathbf{a}, h(\mathbf{a})), \tilde{\partial}),$$

$$\{x''_i\}_{i \in \mathcal{I}} \cup \mathbf{y}' \mapsto \{z_{\infty, i}\}_{i \in \mathcal{I}} \cup \mathbf{y}',$$

where  $(\Sigma, \beta, \alpha, z)$  is as given in Section 4.9.1.

□

Recall from Section 4.9.3 that

$$(\widehat{CF}(S, \mathbf{a}, h(\mathbf{a})), \hat{\partial}') = (\widehat{CF}'(S, \mathbf{a}, h(\mathbf{a})), \partial') / \sim$$

is the  $E^1$ -term of the spectral sequence for  $(\widehat{CF}(\Sigma, \beta, \alpha, z), \hat{\partial})$  (viewed as a double complex) in Theorem 4.9.4. (Here we are writing  $\hat{\partial}'$  for the differential of  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$  to distinguish it from the differential  $\hat{\partial}$  of  $\widehat{CF}(\Sigma, \beta, \alpha, z)$ .)

In this paragraph  $\mathbf{y}$  and  $\mathbf{y}'_i$  denote linear combinations of  $2g$ -tuples. By tracing the zigzags in the double complex we obtain the isomorphism

$$\nu : \widehat{HF}(S, \mathbf{a}, h(\mathbf{a})) \xrightarrow{\sim} \widehat{HF}(\Sigma, \beta, \alpha, z),$$

which is defined as follows: Let  $\mathbf{y} \in \widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))$  be a cycle in  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ , i.e.,

$$\partial' \mathbf{y} = \sum_i (\{x_i\} \cup \mathbf{y}'_i + \{x'_i\} \cup \mathbf{y}'_i).$$

Then  $\nu$  maps the equivalence class  $[\mathbf{y}]$  to the equivalence class of

$$\mathbf{y} + \sum_i \{x''_i\} \cup \mathbf{y}'_i + \text{h.o.},$$

where “h.o.” means terms with more  $x''_i$  components. Composing with the map induced by  $\kappa$  in homology, we obtain an isomorphism

$$\tilde{\nu} = \kappa_* \circ \nu : \widehat{HF}(S, \mathbf{a}, h(\mathbf{a})) \xrightarrow{\sim} \widehat{HF}(S, \mathbf{a}, h(\mathbf{a})).$$

### 6.6.2. The map $\tilde{\Phi}$ .

**Definition 6.6.3.** Let  $\bar{\mathcal{J}}_+ \in \bar{\mathcal{J}}_+^{reg}$  which restricts to  $\bar{\mathcal{J}} \in \bar{\mathcal{J}}^{reg}$  above. We define the map  $\tilde{\Phi}$  as follows:

$$\langle \tilde{\Phi}(\{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}'), \gamma \rangle = \#\mathcal{M}_{\bar{\mathcal{J}}_+}^{I=0, n^* \leq |\mathcal{I}|}(\{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}', \gamma).$$

In our analysis of  $\tilde{\Phi}$  we will use balanced coordinates (cf. Section 5.1.2) for  $\bar{N} - \text{int}(N) = D^2 \times (\mathbb{R}/2\mathbb{Z})$ . The Morse-Bott family  $\mathcal{N}$  can be identified with  $\partial D^2$  and we write  $\gamma_\phi$  for the orbit in  $\mathcal{N}$  corresponding to  $e^{i\phi}$ . So far in this paper  $h \in \mathcal{N}$  was an arbitrary point.

**Convention 6.6.4.** From now on we specialize  $h$  so that  $h = \gamma_{\phi_h}$  is generic and  $\phi_h$  is close to  $-\frac{2\pi}{m}$ , where  $m$  is as defined in Section 5.2.2. In particular, the radial ray corresponding to  $\phi_h$  does not lie on the thin wedges from  $\bar{a}_i$  to  $\bar{h}(\bar{a}_i)$  for all  $i$ . There are no restrictions on the orbit  $e$  except that  $e \neq h$ .

**Lemma 6.6.5.**  $\tilde{\Phi}(\{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}') = 0$  if  $\mathcal{I} \neq \emptyset$ .

*Proof.* Let us fix  $\{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}'$  with  $|\mathcal{I}| \geq 1$  and write

$$\mathcal{M}^0(\gamma) := \mathcal{M}_{\bar{\mathcal{J}}_+}^{I=0, n^* \leq |\mathcal{I}|}(\{z_{\infty,i}\}_{i \in \mathcal{I}} \cup \mathbf{y}', \gamma).$$

We claim that  $\mathcal{M}^0(\gamma) = \emptyset$ . Arguing by contradiction, suppose there exists  $\bar{u} \in \mathcal{M}^0(\gamma)$ . Since  $n^*(\bar{u}) \leq |\mathcal{I}|$ , there is a decomposition  $\bar{u} = \bar{u}'' \cup \bar{u}'''$ , where  $I(\bar{u}'') = I(\bar{u}''') = 0$ ,  $\bar{u}''$  has image in  $\bar{W}_+ - \text{int}(W_+)$ , and  $\bar{u}'''$  has image in  $W_+$ . Then  $\bar{u}''$  is a curve from  $z_\infty$  to  $h$  since  $I(\bar{u}'') = 0$ . The proof of the nonexistence of  $\bar{u}''$  is modeled on the proof of [CGH1, Lemma 10.2.2]. By [We1, Section 4.2],  $\mathbb{R} \times (\bar{N} - N - \delta_0)$  is foliated by finite energy cylinders  $Z_{s_0, \phi_0}$ ,  $(s_0, \phi_0) \in \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ , from  $\delta_0$  to  $\gamma_{\phi_0}$  such that:

- the image of  $Z_{s_0, \phi_0}$  under the projection  $\pi_{\bar{N}} : \mathbb{R} \times \bar{N} \rightarrow \bar{N}$  is the open annulus  $\{\phi = \phi_0, 0 < \rho < 1\}$ ;



- $Z_{s_0+s_1, \phi_0}$  is the  $s_1$ -translate of  $Z_{s_0, \phi_0}$ .

We then set  $Z_{s, \phi}^+ = Z_{s, \phi} \cap \overline{W}_+$  and examine the intersections  $Z_{s, \phi}^+ \cap \overline{u}''$ . Observe that  $K = \pi_N(Z_{s, \phi}^+) \cap \pi_N(\overline{u}'') \neq \emptyset$  for a suitable choice of  $\phi$  which is close to but not equal to  $\phi_h$ ; this is possible by the positioning of  $h$  given by Convention 6.6.4. Hence  $Z_{s_0, \phi}^+ \cap \overline{u}'' \neq \emptyset$  for some  $s_0$ . On the other hand, since  $K$  is compact,  $Z_{s_0+s_1, \phi}^+ \cap \overline{u}'' = \emptyset$  for a sufficiently large  $s_1$ . Finally, since  $Z_{s, \phi}^+$  and  $\overline{u}''$  have no boundary intersections and no intersections near their ends for all  $s \in \mathbb{R}$ , we have a contradiction.  $\square$

**Theorem 6.6.6.**  $\tilde{\Phi}$  is a chain map.

*Proof.* Since  $\Phi$  is a chain map by Theorem 6.2.4, it suffices to verify that

$$\partial_{PFH} \circ \tilde{\Phi}(\{z_{\infty, i}\}_{i \in \mathcal{I}} \cup \mathbf{y}') = \tilde{\Phi} \circ \tilde{\partial}(\{z_{\infty, i}\}_{i \in \mathcal{I}} \cup \mathbf{y}').$$

On the one hand,  $\tilde{\Phi}(\{z_{\infty, i}\}_{i \in \mathcal{I}} \cup \mathbf{y}') = 0$  by Lemma 6.6.5. On the other hand, if  $|\mathcal{I}| = 1$ , then

$$\tilde{\Phi} \circ \tilde{\partial}(\{z_{\infty, i_1}\} \cup \mathbf{y}') = \tilde{\Phi}(\{x_{i_1}\} \cup \mathbf{y}' + \{x'_{i_1}\} \cup \mathbf{y}') = 0;$$

and if  $|\mathcal{I}| > 1$ , then each term of  $\tilde{\partial}(\{z_{\infty, i}\}_{i \in \mathcal{I}} \cup \mathbf{y}')$  contains some copy of  $z_{\infty}$ , and  $\tilde{\Phi}$  maps the term to zero by Lemma 6.6.5. This proves the theorem.  $\square$

A corollary of Lemma 6.6.5 is the following:

**Corollary 6.6.7.**  $\Phi = \tilde{\Phi} \circ \kappa \circ \nu$  on the level of homology.

## 7. THE CHAIN MAP FROM $PFH$ TO $\widehat{HF}$

**7.1. Definition of  $\Psi$ .** Fix the integer  $m \gg 0$  which appears in the definition of the monodromy map  $\bar{h} = \bar{h}_m$  from Section 5.1.2. The condition  $m \gg 0$  will be useful when applying limiting arguments in Sections 7.9 and 7.11.

Suppose  $\overline{\mathcal{J}}_- \in \overline{\mathcal{J}}_-^{reg}$ ,  $\overline{\mathcal{J}}'$  and  $\overline{\mathcal{J}}$  are restrictions of  $\overline{\mathcal{J}}_-$  to the positive and negative ends, and  $\overline{\mathcal{J}}_-^\diamond$  is  $K_{p, 2\delta}$ -regular with respect to  $\overline{\mathbf{m}} = ((0, \frac{3}{2}), z_\infty)$ . We assume that the constants  $\varepsilon, \delta > 0$  that go into the definition of  $\overline{\mathcal{J}}_-^\diamond$  (cf. Section 5.8.4) are arbitrarily small.

**Definition 7.1.1** (Definitions of  $\Psi'$  and  $\Psi$ ).

(1) We define the map

$$\Psi' = \Psi'_{\overline{\mathcal{J}}_-^\diamond}(m, \overline{\mathbf{m}}) : PFC_{2g}(N) \rightarrow \widehat{CF}'(S, \mathbf{a}, h(\mathbf{a}))$$

$$\gamma \mapsto \sum_{\mathbf{y} \in S_{\mathbf{a}, h(\mathbf{a})}} \langle \Psi'(\gamma), \mathbf{y} \rangle \cdot \mathbf{y},$$

where  $\langle \Psi'(\gamma), \mathbf{y} \rangle$  is the mod 2 count of  $\mathcal{M}_{\overline{\mathcal{J}}_-^\diamond}^{I=2, n^*=m}(\gamma, \mathbf{y}; \overline{\mathbf{m}})$ .

(2) We define the map  $\Psi = \Psi_{\overline{\mathcal{J}}_-^\diamond}(m, \overline{\mathbf{m}})$  as the composition

$$PFC_{2g}(N) \xrightarrow{\Psi'} \widehat{CF'}(S, \mathbf{a}, h(\mathbf{a})) \xrightarrow{q} \widehat{CF}(S, \mathbf{a}, h(\mathbf{a})),$$

where  $q$  is the quotient map of chain complexes.

The count  $\langle \Psi'(\gamma), \mathbf{y} \rangle$  is meaningful because of the following theorem:

**Theorem 7.1.2.**  $\mathcal{M}_{\overline{\mathcal{J}}_-^\diamond}^{I=2, n^*=m}(\gamma, \mathbf{y}; \overline{\mathbf{m}})$  is compact.

*Proof.* This follows from Theorems 7.7.3 and 7.11.1(i) and Corollary 7.12.1.  $\square$

Let  $\partial'_{HF}$  be the differential for  $\widehat{CF'}(S, \mathbf{a}, h(\mathbf{a}))$  and let  $\partial_{PFH}$  be the differential for  $PFC_{2g}(N)$ . If  $\mathbf{y}'$  is a  $(2g-1)$ -tuple of chords from  $h(\mathbf{a})$  to  $\mathbf{a}$  such that  $\{x_i\} \cup \mathbf{y}'$  (and also  $\{x'_i\} \cup \mathbf{y}'$ ) is in  $\mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$ , then we define

$$(7.1.1) \quad \tilde{\partial}_1(\{z_\infty\} \cup \mathbf{y}') = \{x_i\} \cup \mathbf{y}' + \{x'_i\} \cup \mathbf{y}',$$

where  $z_\infty = z_{\infty, i}$ .

The proof of the following theorem will occupy the rest of Section 7.

**Theorem 7.1.3.** If  $m \gg 0$ , then

$$(7.1.2) \quad \partial'_{HF} \circ \Psi' + \Psi' \circ \partial_{PFH} = \tilde{\partial}_1 \circ \tilde{\Psi}_0 \circ \tilde{U}_{m-1}.$$

The maps  $\tilde{U}_{m-1}$  and  $\tilde{\Psi}_0$  are defined as follows: Let  $\gamma \in \widehat{\mathcal{O}}_{2g}$  and  $\gamma' \in \widehat{\mathcal{O}}_{2g-p}$ ,  $p = 1, \dots, 2g$ . Then

$$\langle \tilde{U}_{m-1}(\gamma), \delta_0^p \gamma' \rangle = \begin{cases} A, & \text{if } p = 1; \\ 0, & \text{if } p \neq 1, \end{cases}$$

where  $A$  is the count of  $I = 2$  multisections  $\overline{u}$  of  $\overline{W'}$  from  $\gamma$  to  $\delta_0 \gamma'$  which satisfy  $n^*(\overline{u}) = m - 1$  and a certain asymptotic condition near  $\delta_0$  (the precise definition will be given in Section 7.8.3), and

$$\langle \tilde{\Psi}_0(\delta_0 \gamma'), \{z_\infty^p\} \cup \mathbf{y}' \rangle = \begin{cases} B, & \text{if } p = 1; \\ 0, & \text{if } p \neq 1, \end{cases}$$

where  $B$  is the count of degree  $2g - 1$ ,  $I = 0$  almost multisections  $\overline{u}$  of  $\overline{W_-}$  from  $\gamma'$  to  $\mathbf{y}'$  which satisfy  $n^*(\overline{u}) = 0$ , together with the section at infinity  $\sigma_\infty^-$  from  $\delta_0$  to  $z_\infty$ .

The subscripts in  $\tilde{U}_{m-1}$ ,  $\tilde{\Psi}_0$ , and  $\tilde{\partial}_1$  indicate that we are counting curves  $\overline{u}$  satisfying  $n^*(\overline{u}) = m - 1, 0$ , and  $1$ .

Assuming Theorem 7.1.3 for the moment, we have the following:

**Corollary 7.1.4.** The map  $\Psi$  is a chain map.

*Proof.* Similar to the proof of Theorem 6.2.4 and based on Equation (7.1.2). There is one major difference:  $\Psi'$  is not a chain map. However, since

$$\{x_i\} \cup \mathbf{y}' + \{x'_i\} \cup \mathbf{y}' \in \ker q,$$

by composing Equation (7.1.2) with  $q$  we obtain:

$$(7.1.3) \quad \widehat{\partial}_{HF} \circ \Psi + \Psi \circ \partial_{PFH} = 0,$$

where  $\widehat{\partial}_{HF}$  is the differential for  $\widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$ .  $\square$

We can also define a map

$$(7.1.4) \quad \underline{\Psi} : PFH_{2g}(N) \rightarrow \widehat{CF}(S, \mathbf{a}, h(\mathbf{a}))$$

which is  $\mathbb{F}[H_2(M; \mathbb{Z})]$ -linear. As for  $\underline{\Phi}$  in Section 6.4, the key point is the definition of maps

$$\mathfrak{A}_- : H_2(\overline{W}_-, \gamma, \mathbf{y}) \rightarrow H_2(M).$$

The main difference is that  $H_2(\overline{W}_-) \cong H_2(\overline{N}) \cong H_2(M) \oplus \langle \overline{S} \rangle$ . Hence we define  $\mathfrak{A}_-(C) = p((\pi_{\overline{N}})_*[C_{\mathbf{y}} \cup C \cup -C_{\gamma}])$ , where  $\pi_{\overline{N}}$  is the projection  $\overline{W}_- \rightarrow \overline{N}$  and  $p : H_2(\overline{N}) \rightarrow H_2(M)$  is the projection given by  $p = Id - \langle \delta_0, \cdot \rangle [\overline{S}]$ .

**7.2. Outline of proof of Theorem 7.1.3.** In this subsection we outline the proof of Theorem 7.1.3.

Let  $\overline{J}_- \in \overline{\mathcal{J}}_-^{reg}$ , with restrictions  $\overline{J}'$  and  $\overline{J}$  to the positive and negative ends, and let  $\overline{J}_-^\diamond$  be  $K_{p,2\delta}$ -regular with respect to  $\overline{\mathbf{m}}$ . Suppose that the constants  $\varepsilon, \delta > 0$  that go into the definition of  $\overline{J}_-^\diamond$  are arbitrarily small.

We abbreviate:

$$\mathcal{M}^i := \mathcal{M}_{\overline{J}_-^\diamond}^{I=i, n^*=m}(\gamma, \mathbf{y}), \quad \mathcal{M}_{\overline{\mathbf{m}}}^i := \mathcal{M}_{\overline{J}_-^\diamond}^{I=i, n^*=m}(\gamma, \mathbf{y}; \overline{\mathbf{m}}).$$

Let  $\overline{\mathcal{M}}_{\overline{\mathbf{m}}}^i$  be the SFT compactification of  $\mathcal{M}_{\overline{\mathbf{m}}}^i$  and let  $\partial \mathcal{M}_{\overline{\mathbf{m}}}^i = \overline{\mathcal{M}}_{\overline{\mathbf{m}}}^i - \mathcal{M}_{\overline{\mathbf{m}}}^i$ .

**Step 1.** By the SFT compactness theorem (Proposition 7.3.1), a sequence  $\overline{u}_i \in \mathcal{M}_{\overline{\mathbf{m}}}^3$  admits a subsequence which converges to a holomorphic building

$$\overline{u}_\infty = \overline{v}_{-b} \cup \cdots \cup \overline{v}_a,$$

where  $\overline{v}_j$  is a holomorphic map to  $\overline{W}_j = \mathbb{R} \times \overline{N}$  for  $j > 0$ ,  $\overline{W}_0 = \overline{W}_-$  for  $j = 0$ , and  $\overline{W}_j = \overline{W}$  for  $j < 0$ , and the levels  $\overline{W}_{-b}, \dots, \overline{W}_a$  are arranged in order from lowest to highest. As before, we write  $\overline{v}_j = \overline{v}_j' \cup \overline{v}_j''$ , where  $\overline{v}_j'$  branch covers the section at infinity  $\sigma_\infty^*$  and  $\overline{v}_j''$  is the union of irreducible components which do not branch cover  $\sigma_\infty^*$ . Here  $*$  =  $\emptyset$ ,  $'$ , or  $-$ .

There are two cases:  $\overline{v}_0' = \emptyset$  or  $\overline{v}_0' \neq \emptyset$ . The latter case is harder, and will be treated first. By analyzing the two constraints  $n^-(\overline{u}_i) = m$  and  $I(\overline{u}_i) = 3$ , we obtain Theorem 7.7.1, which gives a preliminary list of possibilities for  $\overline{u}_\infty$  when  $\overline{v}_0' \neq \emptyset$ . Many of the possibilities in Theorem 7.7.1 actually do not occur. Theorem 7.11.1 eliminates Cases (2)–(6) from the list. This is done by a finer analysis of the behavior of  $\overline{u}_i$  in the vicinity of the component  $\sigma_\infty^-$  and is similar in spirit to the *layer structures* of Ionel-Parker [IP1, Section 7].

Summarizing, we have:

**Lemma 7.2.1.** *If  $\overline{u}_\infty \in \partial \mathcal{M}_{\overline{\mathbf{m}}}^3$  and  $\overline{v}_0' \neq \emptyset$ , then  $\overline{u}_\infty \in A_1$ , where*

$$A_1 = \coprod_{\delta_0 \gamma', \{z_\infty\} \cup \mathbf{y}'} \left( \mathcal{M}_{\overline{J}'}^{I=2, n^*=m-1}(\gamma, \delta_0 \gamma') \times \mathcal{M}_{\overline{J}_-^\diamond}^{I=0, n^*=0}(\delta_0 \gamma', \{z_\infty\} \cup \mathbf{y}') \right. \\ \left. \times \left( \mathcal{M}_{\overline{J}}^{I=1, n^*=1}(\{z_\infty\} \cup \mathbf{y}', \mathbf{y}) / \mathbb{R} \right) \right),$$

if  $\mathbf{y} = \{x_i\} \times \mathbf{y}'$  or  $\{x'_i\} \times \mathbf{y}'$  and  $A_1 = \emptyset$  otherwise. Here we have omitted the potential contributions of connector components for simplicity.

Here the only remaining Case (1) in Theorem 7.7.1 corresponds to  $A_1$ .

**Step 2.** We now glue the triples  $(\bar{v}_1, \bar{v}_0, \bar{v}_{-1})$  in  $A_1$ , subject to the constraint  $\bar{m}$ . This gluing accounts for the term  $\tilde{\partial}_1 \circ \tilde{\Psi}_0 \circ \tilde{U}_{m-1}$  and is a bit involved. Let us abbreviate

$$\begin{aligned}\mathcal{M}_1 &:= \mathcal{M}_{\bar{J}'}^{I=2, n^*=m-1}(\gamma, \delta_0 \gamma'); \\ \mathcal{M}'_1 &:= \mathcal{M}_{\bar{J}'}^{I=2, n^*=m-1, f_{\delta_0}}(\gamma, \delta_0 \gamma'); \\ \mathcal{M}_0 &:= \mathcal{M}_{\bar{J}_-^\diamond}^{I=0, n^*=0}(\delta_0 \gamma', \{z_\infty\} \cup \mathbf{y}'); \\ \mathcal{M}_{-1} &:= \mathcal{M}_{\bar{J}}^{I=1, n^*=1}(\{z_\infty\} \cup \mathbf{y}', \mathbf{y}),\end{aligned}$$

where  $\mathbf{y} = \{x_i\} \cup \mathbf{y}''$  or  $\{x'_i\} \cup \mathbf{y}''$ . Here  $f_{\delta_0}$  is a nonzero normalized asymptotic eigenfunction of  $\delta_0$  at the negative end which, used as a modifier, stands for “the normalized asymptotic eigenfunction at the negative end  $\delta_0$  is  $f_{\delta_0}$ .” See Section 7.8 for more details on asymptotic eigenfunctions.

We first observe that  $\bar{v}_0 = \bar{v}'_0 \cup \bar{v}''_0 \in \mathcal{M}_0$  is regular:  $\bar{v}'_0$  is regular by Lemma 5.8.9 and  $\bar{v}''_0$  is regular since  $\bar{J}_- \in \bar{J}_-^{reg}$  and  $\bar{J}_-^\diamond$  is  $(\varepsilon, U)$ -close to  $\bar{J}_-$ . The moduli spaces  $\mathcal{M}_1$  and  $\mathcal{M}_{-1}$  are also regular since  $\bar{J}_-$  is regular. Hence we can glue triples  $([\bar{v}_1], \bar{v}_0, [\bar{v}_{-1}])$  in  $(\mathcal{M}_1/\mathbb{R}) \times \mathcal{M}_0 \times (\mathcal{M}_{-1}/\mathbb{R})$ . More precisely, consider the gluing parameter space

$$(7.2.1) \quad \mathfrak{P} := \coprod_{\delta_0 \gamma', \{z_\infty\} \cup \mathbf{y}'} \mathfrak{P}_{\delta_0 \gamma', \{z_\infty\} \cup \mathbf{y}'},$$

where

$$(7.2.2) \quad \mathfrak{P}_{\delta_0 \gamma', \{z_\infty\} \cup \mathbf{y}'} = (5r, \infty)^2 \times (\mathcal{M}_1/\mathbb{R}) \times \mathcal{M}_0 \times (\mathcal{M}_{-1}/\mathbb{R}),$$

$0 < h < 1$  and  $r \gg 1/h$  are gluing constants, and  $\mathcal{M}_1, \mathcal{M}_0, \mathcal{M}_{-1}$  correspond to the pair  $(\delta_0 \gamma', \{z_\infty\} \cup \mathbf{y}')$ . We may assume that all the multiplicities of  $\gamma'$  are 1, since the Hutchings-Taubes gluing of branched covers is essentially independent of the present gluing problem.

Next we introduce the extended moduli space

$$\mathcal{M}^{ext} := \mathcal{M}_{\bar{J}_-}^{I=3, n^*=m, ext}(\gamma, \mathbf{y}),$$

where  $\mathbf{y} = \{x_i\} \cup \mathbf{y}''$  or  $\{x'_i\} \cup \mathbf{y}''$ . Here the modifier *ext* means that  $\bar{u} : (\dot{F}, j) \rightarrow (\bar{W}_-, \bar{J}_-)$  is a multisection which maps all the connected components of  $\partial \dot{F}$  but one to a different  $L_{\bar{a}_i}^-$  and the last connected component to some  $L_{\bar{a}_i \cup \bar{a}_{i,j}}^-$ .

There is a gluing map

$$G : \mathfrak{P} \rightarrow \mathcal{M}^{ext}, \quad \mathfrak{d} = (T_\pm, \bar{v}_1, \bar{v}_0, \bar{v}_{-1}) \mapsto \bar{u}(\mathfrak{d}),$$

which is a diffeomorphism onto its image for  $r \gg 0$ . Here  $T_\pm$  is shorthand for the pair  $T_+, T_-$ . The map  $G$  is defined in a manner similar to that of Section 6.5.2 and

is described in more detail in Section 7.13.3. Let  $r_0 \gg 0$ , let  $\mathfrak{P}_{(r_0)} \subset \mathfrak{P}$  be the subset  $\{T_+ \geq r_0\}$  and let

$$\mathcal{M}_{\overline{\mathfrak{m}}}^{3,(r_0)} := \mathcal{M}_{\overline{\mathfrak{m}}}^3 - G(\mathfrak{P}_{(r_0)})$$

be the truncated moduli space. For generic  $r_0 \gg 0$ ,

$$\partial_0 \mathcal{M}_{\overline{\mathfrak{m}}}^{3,(r_0)} := G(\partial \mathfrak{P}_{(r_0)}) \cap \mathcal{M}_{\overline{\mathfrak{m}}}^3$$

is a transverse intersection.

The following theorem is proved in Section 7.13:

**Theorem 7.2.2.** *For generic  $r_0 \gg 0$ ,*

(7.2.3)

$$\#(\partial_0 \mathcal{M}_{\overline{\mathfrak{m}}}^{3,(r_0)} \cap G(\mathfrak{P}_{\delta_0 \gamma', \{z_\infty\} \cup \mathbf{y}'})) \equiv \#(\mathcal{M}'_1 / \mathbb{R}) \cdot \#\mathcal{M}_0 \cdot \#(\mathcal{M}_{-1} / \mathbb{R}) \pmod{2}.$$

Hence the contributions from  $A_1$  account for the term  $\tilde{\partial}_1 \circ \tilde{\Psi}_0 \circ \tilde{U}_{m-1}$ .

**Step 3.** Let  $\partial_1 \mathcal{M}_{\overline{\mathfrak{m}}}^{3,(r_0)} = \partial \mathcal{M}_{\overline{\mathfrak{m}}}^{3,(r_0)} - \partial_0 \mathcal{M}_{\overline{\mathfrak{m}}}^{3,(r_0)}$ . The following lemma is proved in Section 7.12.

**Lemma 7.2.3.**  $\partial_1 \mathcal{M}_{\overline{\mathfrak{m}}}^{3,(r_0)} \subset A_2 \sqcup A_3$ , where

$$\begin{aligned} A_2 &= \coprod_{\mathbf{y}'' \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}} \left( \mathcal{M}_{\overline{\mathcal{J}}_-}^{I=2, n^*=m}(\gamma, \mathbf{y}''; \overline{\mathfrak{m}}) \times \left( \mathcal{M}_{\overline{\mathcal{J}}}^{I=1, n^*=0}(\mathbf{y}'', \mathbf{y}) / \mathbb{R} \right) \right); \\ A_3 &= \coprod_{\gamma' \in \widehat{\mathcal{O}}_{2g}} \left( \left( \mathcal{M}_{\overline{\mathcal{J}'}}^{I=1, n^*=0}(\gamma, \gamma') / \mathbb{R} \right) \times \mathcal{M}_{\overline{\mathcal{J}}_-}^{I=2, n^*=m}(\gamma', \mathbf{y}; \overline{\mathfrak{m}}) \right). \end{aligned}$$

Here we have omitted the potential contributions of connector components for simplicity.

On the other hand, gluing the pairs in  $A_2$  (resp.  $A_3$ ) using the Hutchings-Taubes gluing theorem implies that  $A_2 \cup A_3 \subset \partial_1 \mathcal{M}_{\overline{\mathfrak{m}}}^{3,(r_0)}$  and accounts for the term  $\partial'_{HF} \circ \Psi'$  (resp.  $\Psi' \circ \partial_{PFH}$ ). This proves Theorem 7.1.3.

*Organization of Section 7.* After some preliminaries (Sections 7.3 and 7.4), the signed intersection numbers  $n^*(\overline{u})$  are analyzed in Sections 7.5 and 7.6. Theorem 7.7.1 is then proved in Section 7.7. We then discuss asymptotic eigenfunctions in Section 7.8, rescaling in Section 7.9, and the “involution lemmas” in Section 7.10, on our way to proving Theorem 7.11.1 in Section 7.11. Lemma 7.2.3 is proved in Section 7.12. Finally, the gluing is discussed in Section 7.13.

### 7.3. SFT compactness.

**Proposition 7.3.1.** *Let  $\overline{\mathcal{J}}_- \in \overline{\mathcal{J}}_-$  and let  $\overline{u}_i : (\dot{F}_i, j_i) \rightarrow (\overline{W}_-, \overline{\mathcal{J}}_-)$ ,  $i \in \mathbb{N}$ , be a sequence of  $\overline{W}_-$ -curves from  $\gamma$  to  $\mathbf{y}$ . Then there is a subsequence which converges in the sense of SFT to a level  $a + b + 1$  holomorphic building*

$$\overline{u}_\infty = \overline{v}_{-b} \cup \cdots \cup \overline{v}_a,$$

where  $\bar{v}_j$  is a holomorphic map to  $\bar{W}_j = \mathbb{R} \times \bar{N}$  for  $j > 0$ ,  $\bar{W}_0 = \bar{W}_-$  for  $j = 0$ , and  $\bar{W}_j = \bar{W}$  for  $j < 0$ , and the levels  $\bar{W}_{-b}, \dots, \bar{W}_a$  are arranged in order from lowest to highest. The same also holds when  $\{\bar{u}_i\}$  is a sequence of  $(\bar{W}_-, \bar{J}_-^\diamond)$ -curves.

*Proof.* The  $\omega$ -area bound for the  $\bar{W}_-$  case is similar to that of the  $W_+$  case. We take the difference of two  $\bar{W}_-$ -curves  $\bar{u}_1$  and  $\bar{u}_2$  from  $\gamma$  to  $\mathbf{y}$  to obtain  $Z \in H_2(\bar{W}_-)$ . Since they both intersect  $\sigma_\infty^-$  once,  $Z$  can be represented by a surface  $\subset W_-$ , and hence can be viewed as a class in  $H_2(N)$ . The zero flux condition implies that the  $\omega$ -area of  $Z$  is zero.

We then use the Gromov-Taubes compactness theorem as before to extract a subsequence  $\bar{u}_i$  which converges in the sense of currents to a holomorphic building. After passing to a subsequence, we may assume that the homology class  $[\bar{u}_i] \in H_2(\bar{W}_-, Z_{\gamma, \mathbf{y}})$  is fixed for all  $i$ . Once the homology class  $[\bar{u}_i]$  is fixed, we use the relative adjunction formula in the same way as in Lemma 6.1.1 to obtain a bound on  $\chi(\dot{F}_i)$ .

Finally, we use the SFT compactness theorem as in Proposition 6.1.2 to obtain a level  $a + b + 1$  holomorphic building  $\bar{u}_\infty$ ; the proof carries over even though  $L_{\bar{\mathbf{a}}}^-$  is a singular Lagrangian submanifold. (We still are left with the task of analyzing the limit more precisely.)  $\square$

**7.4. Novelty of the  $\bar{W}_-$  case.** For the next several subsections (Sections 7.4–7.6),  $\bar{J}_- \in \bar{\mathcal{J}}_-$  and  $\bar{u}_\infty = \bar{v}_{-b} \cup \dots \cup \bar{v}_a$  is the SFT limit of a sequence  $\bar{u}_i : \dot{F} \rightarrow (\bar{W}_-, \bar{J}_-)$  of  $\bar{W}_-$ -curves from  $\gamma$  to  $\mathbf{y}$ .

*Remark 7.4.1.* The same conclusions hold when  $\{\bar{u}_i\}$  is a sequence of  $(\bar{W}_-, \bar{J}_-^\diamond)$ -curves, but we treat  $(\bar{W}_-, \bar{J}_-)$ -curves for simplicity.

We assume that we have already passed to a subsequence so that the domain of  $\bar{u}_i$  is independent of  $i$ . Let  $D^2 = \{\rho \leq 1\} \subset \bar{S}$  with polar coordinates  $(\rho, \phi)$  and let  $z_\infty = (0, 0)$ , as in Section 5.1.2.

The main novelty of the  $\bar{W}_-$  case, as opposed to the  $W_+$  case, is that there may exist a sequence of points  $z_i \in \partial \dot{F}$  such that  $\bar{u}_i(z_i)$  approaches the section at infinity  $\sigma_\infty^-$  as  $i \rightarrow \infty$ . On the HF end, there is an extra Reeb chord  $[0, 1] \times \{z_\infty\}$  which connects any  $\bar{h}(\bar{a}_k)$  to any  $\bar{a}_l$ . On the ECH end, there is an extra closed orbit  $\delta_0 = \{\rho = 0\} \subset \bar{N}$ . Notice that the only orbit in  $\bar{N} - N$  which intersects  $\bar{S} \times \{0\}$  at most  $2g$  times is  $\delta_0$ . Therefore the levels  $\bar{v}_j$  may be asymptotic to  $[0, 1] \times \{z_\infty\}$  or  $\delta_0$  at the ends.

By the SFT compactness argument from Proposition 6.1.2, the map  $\bar{v}_0 : \dot{F}_0 \rightarrow \bar{W}_-$  satisfies one of the following on a neighborhood of a boundary puncture  $p$  of  $\dot{F}_0$ :

- (i)  $\bar{v}_0$  takes the neighborhood of  $p$  to a trivial strip over a Reeb chord (this includes  $[0, 1] \times \{z_\infty\}$ );
- (ii)  $p$  is a removable puncture and the extension  $\bar{v}_0 : \dot{F}_0 \cup \{p\} \rightarrow \bar{W}_-$  satisfies  $\bar{v}_0(p) \in L_{\bar{a}_k}^-$ ; or

- (iii)  $p$  is a removable puncture and the extension  $\bar{v}_0 : \dot{F}_0 \cup \{p\} \rightarrow \bar{W}_-$  satisfies  $\bar{v}_0(p) \in L_{\bar{a}}^- - L_{\bar{a}}^-$ .

We will use the convention that  $\bar{v}_0$  takes each component of  $\partial \dot{F}_0$  to a single  $L_{\bar{a}_k}^-$ ; hence removable punctures of type (ii) are eliminated.

An analogous convention will be used for  $\bar{v}_j : \dot{F}_j \rightarrow \bar{W}$ ,  $j < 0$ , and a removable puncture of type (iii) in this case is:

- (iii')  $p$  is a removable puncture and the extension  $\bar{v}_j : \dot{F}_j \cup \{p\} \rightarrow \bar{W}$  satisfies  $\bar{v}_j(p) \in \mathbb{R} \times \{0, 1\} \times \{z_\infty\}$ .

**Definition 7.4.2.** Let  $\tilde{v}$  be an irreducible component of  $\bar{v}_j : \dot{F}_j \rightarrow \bar{W}$  or  $\bar{W}_j$ ,  $j \leq 0$ , which does not branch cover  $\sigma_\infty^*$ . Then a boundary puncture  $p$  of  $\tilde{v}$  is *removable at  $z_\infty$*  if it is of type (iii).

**7.5. Intersection numbers.** In order to analyze the SFT limit  $\bar{u}_\infty$ , we use the intersection numbers  $n^*(\bar{u}_i)$  and  $n^*(\bar{v}_j)$  to constrain the behavior of holomorphic maps which are asymptotic to  $\delta_0$  or to  $[0, 1] \times \{z_\infty\}$ .

We briefly recall the definition of the intersection numbers  $n^*(\bar{u})$  given in Section 5. Let  $\rho_0 > 0$  be sufficiently small. Consider the torus  $T_{\rho_0} = \{\rho = \rho_0\} \subset \bar{N}$ , oriented as the boundary of  $\{\rho \leq \rho_0\}$ . We take an oriented identification  $T_{\rho_0} \simeq \mathbb{R}^2/\mathbb{Z}^2$  such that the meridian has slope 0 and the closed orbits of  $\bar{R}_0$  on  $T_{\rho_0}$  have slope  $m$ . We pick a closed orbit  $\delta_0^\dagger \subset T_{\rho_0}$  and consider the parallel sections  $(\sigma_\infty^*)^\dagger$  determined by  $\delta_0^\dagger$ . We assume we have chosen  $\delta_0^\dagger$  so that  $(\sigma_\infty^*)^\dagger$  is disjoint from the relevant Lagrangian submanifold. Then  $n^*(\bar{u}) = \langle \bar{u}, (\sigma_\infty^*)^\dagger \rangle$ .

Since  $n^-(\bar{u}_i) = m \gg 2g$  for an  $\bar{W}_-$ -curve  $\bar{u}_i$ , we have

$$(7.5.1) \quad \sum_{j=-b}^a n^*(\bar{v}_j) = m.$$

**Lemma 7.5.1** (Intersection with  $\bar{v}_j$ ,  $j > 0$ ). *Suppose  $j > 0$ . Then the following hold for  $\rho_0 > 0$  sufficiently small:*

- (1) *If  $\bar{v}_j''$  has a positive end  $\mathcal{E}_+$  which converges to  $\delta_0^p$ , then*

$$(7.5.2) \quad \langle \mathcal{E}_+, (\sigma'_\infty)^\dagger \rangle \geq p.$$

- (2) *If  $\bar{v}_j''$  has a negative end  $\mathcal{E}_-$  which converges to  $\delta_0^p$ , then*

$$(7.5.3) \quad \langle \mathcal{E}_-, (\sigma'_\infty)^\dagger \rangle \geq m - p.$$

*If  $\bar{v}_j''$  has multiple ends at covers of  $\delta_0$ , then their contributions to  $n^*(\bar{v}_j)$  are summed.*

*Proof.* Let  $\mathcal{E}_+$  be a positive end which converges to  $\delta_0^p$ . Let  $\pi_{\bar{N}} : \mathbb{R} \times \bar{N} \rightarrow \bar{N}$  be the projection onto the second factor. Provided  $\rho_0$  is sufficiently small,  $\pi_{\bar{N}}(\mathcal{E}_+) \cap T_{\rho_0}$  determines a homology class  $(q, p) \in H_1(T_{\rho_0}) \simeq \mathbb{R}^2/\mathbb{Z}^2$ . One can easily check that

$$\langle \mathcal{E}_+, (\sigma'_\infty)^\dagger \rangle = \det((1, m), (q, p)) = p - qm.$$

Since  $\langle \mathcal{E}_+, (\sigma'_\infty)^\dagger \rangle > 0$  by the positivity of intersections in dimension four and  $m \gg p$ , we must have  $q \leq 0$ . We then obtain:

$$\langle \mathcal{E}_+, (\sigma'_\infty)^\dagger \rangle \geq p.$$

Let  $\mathcal{E}_-$  be a negative end which converges to  $\delta_0^p$ . As above,  $\pi_{\overline{N}}(\mathcal{E}_-) \cap T_{\rho_0}$  determines a homology class  $(q, -p) \in H_1(T_{\rho_0}) \simeq \mathbb{R}^2/\mathbb{Z}^2$  such that

$$\langle \mathcal{E}_-, (\sigma'_\infty)^\dagger \rangle = \det((1, m), (q, -p)) = -p - qm.$$

Since  $\langle \mathcal{E}_-, (\sigma'_\infty)^\dagger \rangle > 0$  by the positivity of intersections in dimension four and  $m \gg p$ , we must have  $q \leq -1$ . We then obtain:

$$\langle \mathcal{E}_-, (\sigma'_\infty)^\dagger \rangle \geq m - p.$$

Finally, if  $\overline{v}_j''$  has multiple ends at covers of  $\delta_0$ , then the total intersection of  $\overline{v}_j''$  with  $(\sigma'_\infty)^\dagger$  is bounded below by the sum of the contributions of each end.  $\square$

Next we consider  $\overline{v}_j$  when  $j < 0$ . Let  $\pi_{\overline{S}} : \mathbb{R} \times [0, 1] \times \overline{S} \rightarrow \overline{S}$  be the projection along the Hamiltonian vector field  $\partial_t$ . Also let  $R_{\phi_0}$  be the radial ray  $\{\phi = \phi_0\} \subset D_\varepsilon^2 = \{\rho \leq \varepsilon\}$ , where  $\varepsilon > 0$  is small. Under the projection  $\pi_{\overline{S}}$ , each positive end  $\mathcal{E}_+$  of  $\overline{v}_j''$  which limits to  $[0, 1] \times \{z_\infty\}$  maps to a sector  $\mathfrak{S}(\mathcal{E}_+)$  of  $D_\varepsilon^2 = \{\rho \leq \varepsilon\}$  going in the counterclockwise direction from  $R_{\phi_1} \subset \overline{a}_{i_1}$  to  $R_{\phi_2} \subset \overline{h}(\overline{a}_{i_2})$ . The sector  $\mathfrak{S}(\mathcal{E}_+)$  is a *thin wedge* if  $i_1 = i_2$  and the angle is  $\frac{2\pi}{m}$ . Similarly, each negative end  $\mathcal{E}_-$  of  $\overline{v}_j''$  which limits to  $[0, 1] \times \{z_\infty\}$  maps to a sector  $\mathfrak{S}(\mathcal{E}_-)$  going counterclockwise from  $R_{\phi_1} \subset \overline{h}(\overline{a}_{i_1})$  to  $R_{\phi_2} \subset \overline{a}_{i_2}$ .

**Lemma 7.5.2** (Intersection with  $\overline{v}_j$ ,  $j < 0$ ). *Suppose  $j < 0$ . Then the following hold for  $\rho_0 > 0$  sufficiently small:*

(1) *If  $\overline{v}_j''$  has a positive end  $\mathcal{E}_+$  which converges to  $[0, 1] \times \{z_\infty\}$ , then*

$$(7.5.4) \quad \langle \mathcal{E}_+, \sigma_\infty^\dagger \rangle \geq 1,$$

*and the relevant sector is a thin wedge if and only if equality holds. Moreover, if the sector is not a thin wedge, then*

$$\langle \mathcal{E}_+, \sigma_\infty^\dagger \rangle > 2g.$$

(2) *If  $\overline{v}_j''$  has a negative end  $\mathcal{E}_-$  which converges to  $[0, 1] \times \{z_\infty\}$ , then*

$$(7.5.5) \quad \langle \mathcal{E}_-, \sigma_\infty^\dagger \rangle > 2g.$$

*Proof.* Suppose  $\rho_0 < \varepsilon$ . The restriction of  $\delta_{\rho_0}$  to  $[0, 1] \times \overline{S}$  consists of  $m$  Reeb arcs  $[0, 1] \times \{\phi_l\}$  on  $\{\rho = \rho_0\}$ , where  $0 \leq l < m$  and  $\phi_l = \phi_0 + l(2\pi/m)$ . Let  $\mathcal{E}_+$  (resp.  $\mathcal{E}_-$ ) be a positive (resp. negative) end of  $\overline{v}_j''$  which converges to  $[0, 1] \times \{z_\infty\}$  and let  $\mathfrak{S}(\mathcal{E}_\pm)$  be the corresponding sector in  $D_\varepsilon^2$ .

By the assumptions on the  $\overline{a}_i$  given in Section 5.2.2, the angles of the thin wedges are  $\frac{2\pi}{m}$  and the other sectors of  $D_\varepsilon^2 - \cup_i \overline{a}_i - \cup_i \overline{h}(\overline{a}_i)$  have angles greater than  $\frac{2\pi(2g)}{m}$ . This implies that:

- (i)  $\langle \mathcal{E}_+, \sigma_\infty^\dagger \rangle \geq 1$ ;
- (ii)  $\langle \mathcal{E}_+, \sigma_\infty^\dagger \rangle = 1$  if and only if  $\mathfrak{S}(\mathcal{E}_+)$  is a thin wedge; and



(iii)  $\langle \mathcal{E}_-, \sigma_\infty^\dagger \rangle > 2g$ .

The lemma follows.  $\square$

Similarly, we have the following lemma, whose proof is the same as those of Lemmas 7.5.1 and 7.5.2 and will be omitted.

**Lemma 7.5.3** (Intersection with  $\bar{v}_0$ ). *The following hold for  $\rho_0 > 0$  sufficiently small:*

(1) *If  $\bar{v}_0''$  has a positive end  $\mathcal{E}_+$  which converges to  $\delta^p$ , then*

$$(7.5.6) \quad \langle \mathcal{E}_+, (\sigma_\infty^-)^\dagger \rangle \geq p.$$

(2) *If  $\bar{v}_0''$  has a negative end  $\mathcal{E}_-$  which converges to  $[0, 1] \times \{z_\infty\}$ , then*

$$(7.5.7) \quad \langle \mathcal{E}_-, (\sigma_\infty^-)^\dagger \rangle > 2g.$$

**Lemma 7.5.4.** *Consider  $\bar{v}_j'' : \dot{F}_j'' \rightarrow \bar{W}_j$ ,  $j \leq 0$ . If  $p \in \partial F_j''$  is a boundary puncture which is removable at  $z_\infty$ , then, for any sufficiently small neighborhood  $N(p) \subset F_j''$ , there exists  $\rho_0 > 0$  such that  $\langle \bar{v}_j''(N(p)), (\sigma_\infty^*)^\dagger \rangle \geq k_0 - 1 > 2g$ .*

Here the constant  $k_0$  is as given in Section 5.2.2.

*Proof.* We will prove the case  $j < 0$ , leaving  $j = 0$  for the reader. Let  $p$  be a removable boundary puncture of  $\bar{v}_j''$  which maps to a point on  $\mathbb{R} \times \{1\} \times \{z_\infty\}$  (without loss of generality). We consider the projection  $\pi_{\bar{S}} : \mathbb{R} \times [0, 1] \times \bar{S} \rightarrow \bar{S}$ . By Definition 5.3.2, the projection  $\pi_{\bar{S}}$  is holomorphic when restricted to  $\pi_{\bar{S}}^{-1}(D_\varepsilon^2)$ , where  $\varepsilon > 0$  is sufficiently small. Hence  $\pi_{\bar{S}} \circ \bar{v}_j''$  is holomorphic when restricted to  $(\pi_{\bar{S}} \circ \bar{v}_j'')^{-1}(D_\varepsilon^2)$  and some sector of  $D_\varepsilon^2 - \bar{a}$  must be contained in  $\text{Im}(\pi_{\bar{S}} \circ \bar{v}_j'')$  because holomorphic maps are open. This in turn shows that

$$\langle \bar{v}_j''(N(p)), (\sigma_\infty^*)^\dagger \rangle \geq k_0 - 1 > 2g$$

by assumption, provided  $\rho_0$  is sufficiently small.  $\square$

**7.6. Some restrictions on  $\bar{u}_\infty$ .** We now present some lemmas in preparation for Theorems 7.7.1 and 7.7.3. Lemmas 7.6.1–7.6.3 give restrictions on  $\bar{u}_\infty$ , which arise from intersection number calculations from Section 7.5, and Lemma 7.6.5 gives a lower bound on the ECH index of the levels  $\bar{v}_j$ .

In what follows, “component” is shorthand for “irreducible component”. We first discuss the “fiber components”, i.e., components  $\tilde{v} : F \rightarrow \bar{W}_j$  of  $\bar{v}_j$  which map to fibers of  $\bar{W}_j$ . There are three types of fiber components:

- (i) ghosts, i.e.,  $\tilde{v}$  is constant;
- (ii) “closed fiber components”, i.e.,  $F$  is closed and  $\tilde{v}$  is nonconstant;
- (iii) “boundary fiber components”, i.e.,  $\partial F \neq \emptyset$  and  $\tilde{v}$  is nonconstant.

If  $\tilde{v}$  is a closed fiber component, then  $\tilde{v}$  is a branched cover of a fiber of  $\bar{W}_j$ . Next let  $\tilde{v} : F \rightarrow \bar{W}_j$  be a boundary fiber component. If  $j < 0$  and  $\tilde{v}$  maps to  $\pi_B^{-1}(p)$ ,  $p \in \partial B$ , then  $\tilde{v}(F) \supset \pi_B^{-1}(p) - A$ , where  $A = L_{\bar{a}}$  or  $L_{\bar{h}(\bar{a})}$ . Similarly, if  $j = 0$  and  $\tilde{v}$  maps to  $\pi_{B_-}^{-1}(p)$ ,  $p \in \partial B_-$ , then  $\tilde{v}(F) \supset \pi_{B_-}^{-1}(p) - L_{\bar{a}}^-$ .

**Lemma 7.6.1.** *The only possible non-ghost fiber component of  $\bar{u}_\infty$  is a closed component  $\bar{\pi}_{B_-}^{-1}(p)$  which passes through  $\bar{m}$ . There is at most one such component.*

*Proof.* Let  $\tilde{v}$  be a non-ghost fiber component. Then  $\tilde{v}$  is a  $d$ -fold branched cover of a fiber if it is a closed fiber component and a  $d$ -fold branched cover of a fiber cut along  $\bar{a}$  (or  $\bar{h}(\bar{a})$ ) if it is a boundary fiber component. In either case,  $n^*(\tilde{v}) = d \cdot m$ . This implies that  $d = 1$ .

We now prove that  $\tilde{v}$  is a closed fiber component passing through  $\bar{m}$ . Arguing by contradiction, if  $\bar{m} \notin \text{Im}(\tilde{v})$ , then there is a component  $\hat{v}$  of  $\bar{u}_\infty$  such that  $\bar{m} \in \text{Im}(\hat{v})$ . If  $\hat{v}$  is a cover of  $\sigma_\infty^-$ , then there must be some component of  $\bar{v}_j''$  for  $j > 0$ , which has a negative end at some  $\delta_0^p$ . Therefore  $n^-(\bar{v}_j'') > 0$  by Lemma 7.5.1(2). If  $\hat{v}$  is not a cover of  $\sigma_\infty^-$ , then  $\hat{v}$  has a nonzero intersection with  $\sigma_\infty^-$  and hence  $n^-(\hat{v}) > 0$ . In either case we have  $n^*(\bar{u}_\infty) > m$ , which is a contradiction. This proves that  $\bar{m} \in \text{Im}(\tilde{v})$ .

Finally boundary fiber components are eliminated because they project to a point in  $\partial B_-$ , and therefore cannot pass through  $\bar{m}$ .  $\square$

We recall the notation  $\bar{v}_j = \bar{v}_j' \cup \bar{v}_j''$ , where  $\bar{v}_j'$  denotes the union of all components of  $\bar{v}_j$  which cover the section at infinity  $\sigma_\infty^*$  and  $\bar{v}_j''$  denotes the union of all other components of  $\bar{v}_j$ . The covering degree of  $\bar{v}_j'$  will always be denoted by  $p_j$ . We also define  $\bar{v}_j^\#$  to be the union of the components of  $\bar{v}_j''$  which are asymptotic to a multiple of  $\delta_0$  or  $z_\infty$  at either end and  $\bar{v}_j^\flat$  to be the union of the remaining non-fiber components of  $\bar{v}_j''$ .

**Lemma 7.6.2.** *If  $\bar{v}_0' = \emptyset$ , then, with the exception of ghost components:*

- (1)  $\bar{v}_j' = \emptyset$  and  $\bar{v}_j^\# = \emptyset$  for all  $j$ ;
- (2) no  $\bar{v}_j''$ ,  $j \leq 0$ , has a boundary puncture which is removable at  $z_\infty$ ;
- (3) every level  $\bar{v}_j$ ,  $j \neq 0$ , has image inside  $W'$  or  $W$ ; and
- (4)  $\bar{v}_0$  is a  $\bar{W}_-$ -curve or a degenerate  $\bar{W}_-$ -curve, i.e.,  $\bar{v}_0$  is the union of a  $W_-$ -curve and a fiber  $\bar{\pi}_{B_-}^{-1}(p)$  which passes through  $\bar{m}$ .

*Proof.* If  $\bar{v}_0' = \emptyset$ , then  $\bar{v}_0 = \bar{v}_0''$ . Since  $\bar{u}_i$  passes through  $\bar{m}$  for all  $i$ , the level  $\bar{v}_0 : \dot{F}_0 \rightarrow \bar{W}_-$  must also pass through  $\bar{m}$ . By Lemma 5.4.10,<sup>11</sup>  $n^-(\bar{v}_0) \geq m$  because  $\bar{m} \in \sigma_\infty^-$ . Equation (7.5.1) and the nonnegativity of  $n^*$  then imply that  $n^-(\bar{v}_0) = m$  and  $n^*(\bar{v}_j) = 0$  for  $j \neq 0$ .

- (1) If  $\bar{v}_j' \neq \emptyset$  or  $\bar{v}_j^\# \neq \emptyset$ , then at least one of Equations (7.5.3)–(7.5.6) applies and  $\sum_{j \neq 0} n^*(\bar{v}_j^\#) > 0$ .
- (2) This follows from  $n^-(\bar{v}_0) = m$  and Lemma 7.5.4.
- (3) Since  $n^*(\bar{v}_j) = 0$  for  $j \neq 0$ , (1), combined with Lemma 5.3.13, implies that  $\text{Im}(\bar{v}_j) \subset W'$  if  $j > 0$ , whereas (1) and (2), combined with Lemma 5.3.8, implies that  $\text{Im}(\bar{v}_j) \subset W$  if  $j < 0$ .

<sup>11</sup>More precisely, this follows from the method of proof of Lemma 5.4.6.

- (4) Since  $n^-(\bar{v}_0) = m$ ,  $\bar{v}_0$  intersects  $\sigma_\infty^-$  only at  $\bar{m}$  and the intersection is transverse by Lemma 5.4.10(2). By (1) and (2) and Lemmas 7.6.1 and 5.4.12, if  $\bar{v}_0$  is not a  $\bar{W}_-$ -curve, then it must be a degenerate  $\bar{W}_-$ -curve.

This completes the proof of the lemma.  $\square$

**Lemma 7.6.3.** *If  $\bar{v}'_0 \neq \emptyset$ , then, with the exception of ghost components:*

- (1) *there is only one negative end of  $\bar{v}_j^\sharp$ ,  $j > 0$ , (say  $\bar{v}_{a'}^\sharp$ ) which is asymptotic to a multiple of  $\delta_0$ ;*
- (2)  *$\bar{u}_\infty$  has no fiber components;*
- (3) *no  $\bar{v}_j'$ ,  $j \leq 0$ , has a boundary puncture which is removable at  $z_\infty$ ;*
- (4)  *$\bar{v}_j^\flat$  has image inside  $W$ ,  $W_-$ , or  $W'$ ;*
- (5)  *$\bar{v}_j^\sharp$ ,  $j < 0$ , is a union of thin strips from  $z_\infty$  to some  $x_i$  or  $x'_i$ ;*
- (6)  *$\bar{v}_0^\sharp$  has image inside  $\bar{W}_- - \text{int}(W_-)$  and has  $\delta_0^{r_0}$ , for some  $r_0 > 0$ , at the positive end and some subset of  $\{x_1, \dots, x_{2g}, x'_1, \dots, x'_{2g}\}$  of cardinality  $p$  at the negative end;*
- (7)  *$\bar{v}_j^\sharp$ ,  $0 < j \leq a'$ , has image inside  $\bar{W}' - \text{int}(W')$  and has  $\delta_0^{r_j}$ , for some  $r_j > 0$ , at the positive end and  $h^{r'_j} e^{r''_j}$  with  $r'_j + r''_j = r_j$ , at the negative end.*

*Proof.* (1) By Lemma 7.5.1, each negative end of  $\bar{v}_j^\sharp$ , for  $j > 0$ , which is asymptotic to a multiple of  $\delta_0$  contributes at least  $m - 2g$  to  $\sum_{j=1}^a n^*(\bar{v}_j)$ , where  $m \gg 2g$ , because the total multiplicity of  $\delta_0$  is  $\leq 2g$ . If there are at least two such negative ends of  $\bar{v}_j^\sharp$ , then the total contribution to  $\sum_{j=1}^a n^*(\bar{v}_j)$  is at least  $2(m - 2g) > m$ , which is a contradiction. Suppose  $\bar{v}_{a'}^\sharp$ ,  $0 < a' \leq a$ , has a negative end at a multiple at  $\delta_0$ .

(2) By Lemma 7.6.1, a fiber component of  $\bar{u}_\infty$  is a fiber  $\pi_{B_-}^{-1}(p)$  which passes through  $\bar{m}$  and additionally contributes  $m$  to  $\sum_{j=-b}^a n^*(\bar{v}_j)$ , a contradiction.

(3) By Lemma 7.5.4, a boundary puncture which is removable at  $z_\infty$  additionally contributes  $\geq 2g$  to  $\sum_{j=-b}^a n^*(\bar{v}_j)$ . This is again a contradiction.

(4) By (3),  $\bar{v}_j^\flat$  does not have any boundary punctures which are removable at  $z_\infty$ . Since the ends of  $\bar{v}_j^\flat$  are contained in  $N$  or  $[0, 1] \times S$  and  $n^*(\bar{v}_j^\flat) = 0$ , we conclude that  $\text{Im}(\bar{v}_j^\flat) \subset W$ ,  $W_-$ , or  $W$  by Lemmas 5.3.8, 5.4.12 and 5.3.13.

(5), (6) Let  $p_j = \deg(\bar{v}_j')$  be the covering degree of  $\bar{v}_j'$  over  $\sigma_\infty'$ . Then  $p_0 \leq p_1 \leq \dots \leq p_{a'-1}$  and  $p_{a'} = 0$ . This follows from (1) since a negative end of  $\bar{v}_{j+1}^\sharp$  is required to decrease  $p_j$ .

Let  $p_{a'}^-$  be the multiplicity of  $\delta_0$  at the negative end of  $\bar{v}_{a'}^\sharp$ . The negative end of  $\bar{v}_{a'}^\sharp$  contributes at least  $m - p_{a'}^-$  to  $n^*$  by Equation (7.5.3) and the positive ends of  $\bar{v}_j^\sharp$ , for  $j = 0, \dots, a' - 1$ , give a total contribution of at least  $p_{a'}^- - p_0$  to  $n^*$  by Equations (7.5.2) and (7.5.6). Hence,

$$(7.6.1) \quad \sum_{j=0}^a n^*(\bar{v}_j^\sharp) \geq m - p_0.$$

On the other hand, by Equation (7.5.4), the contributions of the positive ends of  $\bar{v}_j^\sharp$ ,  $j = -b, \dots, -1$ , add up to

$$(7.6.2) \quad \sum_{j=-b}^{-1} n^*(\bar{v}_j^\sharp) \geq p_0.$$

Equations (7.6.1) and (7.6.2) give:

$$(7.6.3) \quad \sum_{j=-b}^a n^*(\bar{v}_j^\sharp) \geq m.$$

Equality holds by Equation (7.5.1). This in turn implies that:

- (i) equality holds in both Equations (7.6.1) and (7.6.2); and
- (ii)  $\bar{v}_j^\sharp$ ,  $j \leq 0$ , has no negative end which limits to  $z_\infty$ .

Since  $p_0 \leq 2g$ , each  $\bar{v}_j^\sharp$ ,  $j < 0$ , must be a union of thin strips by Lemma 7.5.2. This gives (5). Moreover,  $\bar{v}_0^\sharp$  has  $\delta_0^{p_1-p_0}$  at the positive end and no negative ends at  $z_\infty$  by (ii) and

$$(7.6.4) \quad n^-(\bar{v}_0^\sharp) = p_1 - p_0$$

by Lemma 7.5.3 and (i).

In order to prove (6), we consider  $C_\rho = \pi_{\bar{N}}(\bar{v}_0^\sharp) \cap T_\rho$ , where  $0 < \rho < 1$ . The argument is similar to the proof of the blocking lemma in [CGH1], and the presence of the Lagrangian boundary condition for  $\bar{v}_0^\sharp$  does not change the proof in any essential way. Suppose  $\rho_1 > 0$  is small. Let  $\cup_i \mathcal{E}_i$  be the union of positive ends of  $\bar{v}_0^\sharp$  that limit to multiples of  $\delta_0$ , and let  $C'_{\rho_1} = \pi_{\bar{N}}(\cup_i \mathcal{E}_i) \cap T_{\rho_1}$ . Viewing  $C'_{\rho_1}$  as a cycle  $(q, p_1 - p_0) \in H_1(T_{\rho_1})$ , we have

$$0 < p_1 - p_0 \leq \det((1, m), (q, p_1 - p_0)) = p_1 - p_0 - qm.$$

By Equation (7.6.4),  $q = 0$  and  $C_{\rho_1} = C'_{\rho_1}$ , since any intersections besides those coming from  $\cup_i \mathcal{E}_i$  would give some extra contribution to  $n^+(\bar{v}_0^\sharp)$ .

Next consider  $C_{\rho_2}$ , where  $\rho_2 = 1 - \varepsilon$ ,  $\varepsilon > 0$  small. Since there are no ends of  $\bar{v}_0^\sharp$  between  $T_{\rho_1}$  and  $T_{\rho_2}$ , it follows that  $C_{\rho_2} = (0, p_1 - p_0) \in H_1(T_{\rho_2})$ . Negative ends can approach  $x_i$  or  $x'_i$  either from the interior of  $V = \bar{N} - \text{int}(N)$  or from the exterior of  $V$ . A negative end of  $\bar{v}_0^\sharp$  approaching  $x_i$  or  $x'_i$  from the interior contributes  $(0, 1)$  to the homology class of  $C_{\rho_2}$ . This means that at most  $p_1 - p_0$  ends of  $\bar{v}_0^\sharp$  approach  $x_i$  or  $x'_i$  from the interior.

Finally let  $\rho_3 = 1 + \varepsilon$ ,  $\varepsilon > 0$  small, and let  $C'_{\rho_3} = \pi_{\bar{N}}(\tilde{v}) \cap T_{\rho_3}$ , where  $\tilde{v}$  is obtained from  $\bar{v}_0^\sharp$  by truncating the negative ends which limit to  $x_i$  or  $x'_i$  from the exterior of  $V$ . Then  $C'_{\rho_3} = (0, r) \in H_1(T_{\rho_3})$  for some  $r \geq 0$ . We can eliminate the possibility  $r > 0$  by examining the intersection of  $C'_{\rho_3}$  and the Hamiltonian vector field on  $T_{\rho_3}$  and applying the positivity of intersections. Since  $\rho_3$  was arbitrarily close to 1, it follows that the image of  $\bar{v}_0^\sharp$  cannot escape  $\bar{W}_- - \text{int}(W_-)$ , which implies (6).

(7) This is similar to (6) and is left to the reader.  $\square$

*Remark 7.6.4.* One can easily compute that  $I(\bar{v}_0^\sharp) = r_0$  and  $I(\bar{v}_j^\sharp) = r'_j + 2r''_j$  when  $j > 0$ .

**Lemma 7.6.5.** *The ECH index of each level  $\bar{v}_j$ ,  $j = -b, \dots, a$ , is nonnegative if  $\bar{J}_- \in \bar{\mathcal{J}}_-^{reg}$ .*

*Proof.* Recall that the restrictions  $\bar{J}$  and  $\bar{J}'$  are also regular by definition. By [HT1, Proposition 7.15(a)],  $I_{ECH}(\bar{v}_j) \geq 0$  for  $j > 0$  since  $\bar{J}'$  is regular.

*Case  $j < 0$ .* We write  $\bar{v}_j = \bar{v}'_j \cup \bar{v}^\sharp_j \cup \bar{v}^\flat_j$ .

Suppose that  $\bar{v}'_0 = \emptyset$ . Then  $\bar{v}'_j = \emptyset$  and  $\bar{v}^\sharp_j = \emptyset$  for all  $j < 0$  by Lemma 7.6.2, and we are left with  $\bar{v}^\flat_j$ , which is simply-covered and has nonnegative Fredholm index by regularity. By the index inequality (Theorem 4.5.13),  $I(\bar{v}_j) = I(\bar{v}^\flat_j) \geq 0$ .

Next suppose that  $\bar{v}'_0 \neq \emptyset$ . Then  $\bar{v}^\sharp_j$  is a union of thin strips from  $z_\infty$  to some  $x_i$  or  $x'_i$  by Lemma 7.6.3 and each thin strip has ECH index 1. We also have  $I(\bar{v}'_j) = 0$  by Lemma 5.7.12 and  $I(\bar{v}^\flat_j) \geq 0$  by the previous paragraph.

We claim that

$$(7.6.5) \quad I(\bar{v}_j) = I(\bar{v}'_j \cup \bar{v}^\sharp_j \cup \bar{v}^\flat_j) = I(\bar{v}'_j) + I(\bar{v}^\sharp_j) + I(\bar{v}^\flat_j).$$

Note that, although  $\bar{v}'_j$ ,  $\bar{v}^\sharp_j$  and  $\bar{v}^\flat_j$  are disjoint,  $\bar{v}'_j$  and  $\bar{v}^\sharp_j$  are both asymptotic to (a multiple of)  $z_\infty$  at the positive end and the additivity of the ECH indices of  $\bar{v}'_j$  and  $\bar{v}^\sharp_j$  needs to be verified. For that purpose, recall from Section 5.7 that each  $\bar{v}'_j$  comes equipped with data  $\mathcal{D}'_j = ((\mathcal{D}')_j^{to}, (\mathcal{D}')_j^{from})$  at the positive and negative ends, since  $\bar{u}_\infty$  is the limit of the sequence  $\{\bar{u}_i\}$ . The key observation here is that  $(\mathcal{D}')_j^{to} = (\mathcal{D}')_j^{from}$ , since all the components of  $\bar{v}'_{j'}$ ,  $j' < j$ , are thin strips whose data  $(\mathcal{D}'_+)_{j'} = ((\mathcal{D}'_+)_{j'}^{to}, (\mathcal{D}'_+)_{j'}^{from})$  at the positive end satisfies  $(\mathcal{D}'_+)_{j'}^{to} = (\mathcal{D}'_+)_{j'}^{from}$ . Hence we can choose a simultaneous grooming  $\mathfrak{c}^+ = \{c_k^+\}$  for both  $\bar{v}'_j$  and  $\bar{v}^\sharp_j$  at the positive end  $z_\infty$  such that  $c_k^+$  has  $w(c_k^+) = 0$  and connects  $\bar{h}(\bar{a}_{i_k, j_k})$  to  $\bar{a}_{i_k, j_k}$ . If we choose a groomed multivalued trivialization  $\tau$  compatible with  $\mathfrak{c}^+$ , then

$$I_\tau(\bar{v}'_j \cup \bar{v}^\sharp_j) = I_\tau(\bar{v}'_j) + I_\tau(\bar{v}^\sharp_j),$$

which immediately implies Equation (7.6.5).

*Case  $j = 0$ .* Suppose that  $\bar{v}'_0 = \emptyset$ . Then,  $\bar{v}_0$  is a  $\bar{W}_-$ -curve or a degenerate  $\bar{W}_-$ -curve by Lemma 7.6.2. If  $\bar{v}_0$  is a  $\bar{W}_-$ -curve, then it is simply-covered and satisfies  $I(\bar{v}_0) \geq \text{ind}(\bar{v}_0) \geq 0$ . If  $\bar{v}_0$  is a degenerate  $\bar{W}_-$ -curve, then  $\bar{v}_0$  can be written as a union of a fiber  $C$  and a  $W_-$ -curve  $\bar{v}_0^\flat$ . The Fredholm index of  $\bar{v}_0^\flat$  is nonnegative since  $\bar{v}_0^\flat$  is simply-covered and hence is regular. The Fredholm index of  $C$  is given by:

$$\begin{aligned} \text{ind}(C) &= -\chi(C) + 2\langle c_1(TW_-), C \rangle \\ &= (2g - 2) + 2(2 - 2g) = 2 - 2g. \end{aligned}$$

The algebraic intersection number  $\langle C, \bar{v}_0^b \rangle$  is equal to  $2g$  and

$$(7.6.6) \quad \begin{aligned} I(\bar{v}_0) &\geq \text{ind}(C) + \text{ind}(\bar{v}_0^b) + 2\langle C, \bar{v}_0^b \rangle \\ &\geq (2 - 2g) + 0 + 2(2g) = 2g + 2, \end{aligned}$$

by Theorem 5.6.9.

Next suppose that  $\bar{v}'_0 \neq \emptyset$ . We have  $I(\bar{v}'_0) = 0$  by Lemma 5.7.13. Next, by Lemma 7.6.3, if  $\bar{v}_0^\sharp \neq \emptyset$ , then  $\text{Im}(\bar{v}_0^\sharp) \subset \overline{W}_- - \text{int}(W_-)$  and has  $\delta_0^{p_1-p_0}, p_1-p_0 > 0$ , at the positive end and some  $(p_1-p_0)$ -element subset of  $\{x_1, \dots, x_{2g}, x'_1, \dots, x'_{2g}\}$  at the negative end. Hence  $I(\bar{v}_0^\sharp) = p_1 - p_0$  by Lemma 5.7.14. Finally, since  $\text{Im}(\bar{v}_0^b) \subset W_-$  by Lemma 7.6.3,  $\bar{v}_0^b$  is simply-covered and  $I(\bar{v}_0^b) \geq 0$ . The ECH indices of  $\bar{v}'_0$ ,  $\bar{v}_0^\sharp$ , and  $\bar{v}_0^b$  are additive.

This completes the proof of the lemma.  $\square$

**7.7. Compactness theorem.** Let  $\bar{\mathcal{J}}_-^\diamond(\varepsilon, \delta, p)$  be a generic almost complex structure which is  $(\varepsilon, U)$ -close to  $\bar{\mathcal{J}}_- \in \bar{\mathcal{J}}_-^{reg}$ , where  $U$  and  $K_{p,2\delta}$  are as in Convention 5.8.12. We write:

$$\mathcal{M}_{\bar{\mathbf{m}}}^i(\varepsilon, \delta, p) := \mathcal{M}_{\bar{\mathcal{J}}_-^\diamond(\varepsilon, \delta, p)}^{I=i, n^*=m}(\gamma, \mathbf{y}; \bar{\mathbf{m}}).$$

As before,  $\bar{u}_\infty \in \partial \mathcal{M}_{\bar{\mathbf{m}}}^3(\varepsilon, \delta, p)$  is written as  $\bar{u}_\infty = \bar{v}_{-b} \cup \dots \cup \bar{v}_a$  and each  $\bar{v}_j$  is written as  $\bar{v}_j = \bar{v}'_j \cup \bar{v}''_j = \bar{v}_j^\sharp \cup \bar{v}_j^b$ .

We now prove the following compactness theorem, which is basically a consequence of two constraints:  $I(\bar{u}_i) = 3$  and  $n^*(\bar{u}_i) = m$ . The list of possibilities should be viewed as a preliminary list, since we will subsequently eliminate Cases (2)–(6) in Theorem 7.11.1.

**Theorem 7.7.1.** *Let  $\bar{\mathcal{J}}_-^\diamond$  be  $(\varepsilon, U)$ -close to  $\bar{\mathcal{J}}_- \in \bar{\mathcal{J}}_-^{reg}$  and  $K_{p,2\delta}$ -regular with respect to  $\bar{\mathbf{m}}$ , and let  $\bar{u}_\infty \in \partial \mathcal{M}_{\bar{\mathbf{m}}}^3(\varepsilon, \delta, p)$ . If  $\bar{v}'_0 \neq \emptyset$ , then  $\bar{u}_\infty$  contains one of the following subbuildings:*

- (1) A 3-level building consisting of  $\bar{v}_1$  with  $I = 2$  from  $\gamma$  to  $\delta_0\gamma'$ ;  $\bar{v}'_0 = \sigma_\infty^-$ ; and  $\bar{v}_{-1}^\sharp$  which is a thin strip from  $z_\infty$  to  $x_i$  or  $x'_i$ .
- (2) A 3-level building consisting of  $\bar{v}_1$  with  $I = 1$  from some  $\gamma''$  to  $\delta_0\gamma'$ ;  $\bar{v}'_0 = \sigma_\infty^-$ ; and  $\bar{v}_{-1}^\sharp$  which is a thin strip.
- (3) A 4-level building consisting of  $\bar{v}_1$  with  $I = 1$  from  $\gamma$  to  $\delta_0^2\gamma'$ ;  $\bar{v}'_0$  which is a branched double cover of  $\sigma_\infty^-$ ;  $\bar{v}'_{-1} = \sigma_\infty^-$ ; and  $\bar{v}_{-1}^\sharp$  and  $\bar{v}_{-2}^\sharp$  which are both thin strips.
- (4) A 3-level building consisting of  $\bar{v}_1$  with  $I = 1$  from  $\gamma$  to  $\delta_0^2\gamma'$ ;  $\bar{v}'_0$  which is a branched double cover of  $\sigma_\infty^-$ ; and  $\bar{v}_{-1}^\sharp$  which is the union of two thin strips.
- (5) A 3-level building consisting of  $\bar{v}_1$  with  $I = 1$  from  $\gamma$  to  $\delta_0^2\gamma'$ ;  $\bar{v}'_0 = \sigma_\infty^-$ ;  $\bar{v}_0^\sharp$  with  $I = 1$  from  $\delta_0$  to  $x_i$  or  $x'_i$ ; and  $\bar{v}_{-1}^\sharp$  which is a thin strip.

- (6) A 4-level building consisting of  $\bar{v}_2$  with  $I = 1$  from  $\gamma$  to  $\delta_0^2 \gamma'$ ;  $\bar{v}_1' = \sigma_\infty'$ ;  $\bar{v}_1^\#$  with  $I = 1$  which is a cylinder from  $\delta_0$  to  $h$ ;  $\bar{v}_0' = \sigma_\infty^-$ ; and  $\bar{v}_{-1}^\#$  which is a thin strip.

Here we are omitting levels which are connectors.

See Figure 4.

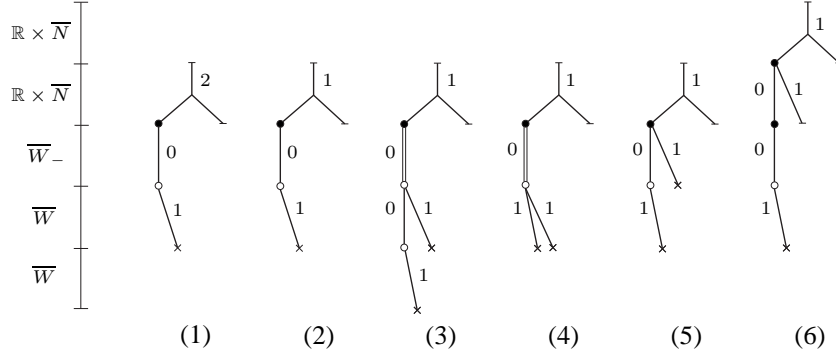


FIGURE 4. Schematic diagrams for the possible types of degenerations. Here  $\bullet$  represents  $\delta_0$ ,  $\circ$  represents  $z_\infty$ , and  $\times$  represents some  $x_i$  or  $x'_i$ . A vertical line indicates a trivial cylinder or a restriction of a trivial cylinder, and a double vertical line indicates a branched double cover of a trivial cylinder or a restriction of a trivial cylinder. The labels on the graphs are ECH indices of each component.

*Proof.* By Proposition 7.3.1,  $\bar{u}_i$  converges in the sense of SFT to a holomorphic building  $\bar{u}_\infty = \bar{v}_a \cup \cdots \cup \bar{v}_{-b}$ . We have three constraints:

- (i)  $\sum_{j=-b}^a I(\bar{v}_j) = 3$ ,
- (ii)  $I(\bar{v}_j) \geq 0$  for all  $j$ , and
- (iii)  $\sum_{j=-b}^a n^*(\bar{v}_j) = m$ .

(i) comes from the additivity of ECH indices, (ii) comes from Lemma 7.6.5, and (iii) comes from Equation (7.5.1).

Suppose that  $\bar{v}_0' \neq \emptyset$ , i.e., we are in the situation of Lemma 7.6.3. We have the following restrictions:

- the top level  $\bar{v}_a$  is nontrivial and satisfies  $I(\bar{v}_a) \geq 1$ ;
- $\cup_{j<0} \bar{v}_j^\#$  consists of  $p_0 \geq 1$  thin strips and contributes  $\sum_{j<0} I(\bar{v}_j^\#) = p_0 \geq 1$  to the total ECH index.

This immediately implies  $p_{a'}^- \leq 2$  because  $p_{a'}^-$  is also the number of nontrivial curves with a positive end at  $\delta_0$  and each of them has ECH index  $I \geq 1$  by Lemma 7.6.3. We also have  $p_0 \leq p_{a'}^-$ , and therefore we can divide the proof into three cases:

- Case I:  $p_{a'}^- = p_0 = 1$ .
- Case II':  $p_{a'}^- = 2$  and  $p_0 = 1$ .
- Case II'':  $p_{a'}^- = 2$  and  $p_0 = 2$ .

*Case I.* In this case  $\bar{v}'_0 = \sigma_\infty^-$  and  $\bar{v}'_{-1}$  is a thin strip. This leaves two possibilities for  $\bar{v}_1$ : either  $I(\bar{v}_1) = 2$  and we are in Case (1), or  $I(\bar{v}_1) = 1$  and we are in Case (2).

*Case II'.* In this case we have  $\bar{v}'_{j_0} \neq 0$  for some  $j_0 \geq 0$ . Since  $I(\bar{v}'_{j_0}) = 1$ ,  $\bar{v}_{-1}$  consists of a single thin strip and there are no other levels with  $j < 0$ . If  $j_0 = 0$  we are in Case (5), and if  $j_0 > 0$  we are in Case (6). In Case (6),  $\bar{v}'_1$  is a cylinder connecting  $\delta_0$  with  $h$  by Lemma 7.6.3 and Remark 7.6.4.

*Case II''.* In this case  $\cup_{j < 0} \bar{v}_j$  consists of two thin strips. If they are on the same level we are in Case (4) and if they are on different levels we are in Case (3).

This completes the proof of Theorem 7.7.1.  $\square$

*Remark 7.7.2.* In Cases (3)–(6), the total number of branch points of  $\cup_{j=-b}^a \bar{v}'_j$  is one, where we are not ignoring connector components that cover  $\sigma_\infty^*$ : Assume without loss of generality that the only nontrivial  $\bar{v}'_j$  is  $\bar{v}'_0$  and that  $\bar{v}'_0$  double covers  $\sigma_\infty^-$ . Let  $\mathfrak{b}$  be the number of branch points of  $\bar{v}'_0$ . Then  $\text{ind}(\bar{v}'_0) = \mathfrak{b} - 1$  by Proposition 5.5.5, the Riemann-Hurwitz formula, and the proof of Lemma 5.8.9. The index inequality, the additivity of the indices, and the condition  $I(\bar{u}_i) = 3$  force  $\mathfrak{b} = 1$ .

The proof of the following theorem is similar and will be omitted.

**Theorem 7.7.3.** *Let  $\bar{J}_-^\diamond$  be  $(\varepsilon, U)$ -close to  $\bar{J}_- \in \bar{\mathcal{J}}_-^{\text{reg}}$  and  $K_{p,2\delta}$ -regular with respect to  $\bar{\mathfrak{m}}$ , and let  $\bar{u}_\infty \in \partial \mathcal{M}_{\bar{\mathfrak{m}}}^2(\varepsilon, \delta, p)$ . If  $\bar{v}'_0 \neq \emptyset$ , then  $\bar{u}_\infty$  contains a 3-level subbuilding consisting of  $\bar{v}_1$  with  $I = 1$  from  $\gamma$  to  $\delta_0 \gamma'$ ;  $\bar{v}'_0 = \sigma_\infty^-$ ; and  $\bar{v}'_{-1}$  which is a thin strip. Here it is possible to have  $I_{ECH} = 0$  connectors in between.*

**7.8. Asymptotic eigenfunctions.** We now collect some facts about the asymptotic operator and asymptotic eigenfunctions, referring the reader to [HWZ1] and [HT2].

**7.8.1. The asymptotic operator.** We study the local behavior of a holomorphic half-cylinder which converges to a degree  $l \geq 1$  multiple cover of  $\delta_0$ , denoted by  $\delta_0^l$ . Let  $D_{\rho_0} = \{\rho \leq \rho_0\} \subset D^2$  with  $\rho_0 > 0$  small (in particular  $\leq \frac{1}{2}$ ) and let  $K = [C, +\infty) \times (\mathbb{R}/2l\mathbb{Z})$  with  $l \in \mathbb{N}$ . We will be using balanced coordinates on  $(\mathbb{R}/2\mathbb{Z}) \times D_{\rho_0}$ ; see Section 5.1.2. A holomorphic half-cylinder which is asymptotic to  $\delta_0^l$  at the positive end restricts to a holomorphic map

$$\bar{u} : K \rightarrow \mathbb{R} \times (\mathbb{R}/2\mathbb{Z}) \times D_{\rho_0},$$

which can be written as:

$$(s, t) \mapsto (s, t, z(s, t)),$$

where  $\lim_{s \rightarrow +\infty} z(s, t) = 0$ .



**Lemma 7.8.1.** *The function  $z : K \rightarrow D_{\rho_0}$  satisfies the equation*

$$(7.8.1) \quad \partial_s z + i\partial_t z + \varepsilon z = 0,$$

where  $\varepsilon = \frac{\pi}{m}$ .

*Proof.* The partial derivatives of  $\bar{u}$  are:  $\partial_s \bar{u} = (1, 0, \partial_s z)$  and  $\partial_t \bar{u} = (0, 1, \partial_t z)$ . Then  $J(\partial_s \bar{u}) = R + (0, 0, i\partial_s z)$ , where  $R$  is the Hamiltonian vector field. We compute that  $R = \partial_t + \varepsilon \partial_\phi$ . Hence  $J(\partial_s \bar{u}) = (0, 1, i\partial_s z + i\varepsilon z)$ . This gives Equation (7.8.1).  $\square$

**Definition 7.8.2.** We define the *asymptotic operator*

$$A_l : L_1^2(\mathbb{R}/2l\mathbb{Z}, \mathbb{C}) \rightarrow L^2(\mathbb{R}/2l\mathbb{Z}, \mathbb{C}),$$

$$f \mapsto -i\partial_t f - \varepsilon f.$$

We remark that the asymptotic operator which appears in [HWZ1] is  $A_l$ , whereas the asymptotic operator in [HT2] is  $-A_l$ . The eigenfunctions of  $A_l$  are the *asymptotic eigenfunctions*, and are given by  $ce^{\pi i n t/l}$ ,  $c \in \mathbb{C}$ ,  $n \in \mathbb{Z}$ , with corresponding eigenvalues  $\frac{\pi n}{l} - \varepsilon$ . An asymptotic eigenfunction  $ce^{\pi i n t/l}$  is said to be *normalized* if  $|c| = 1$ . Let  $E_{\pi n/l - \varepsilon}$  be the eigenspace of  $A_l$  corresponding to the eigenvalue  $\frac{\pi n}{l} - \varepsilon$ .

For a strip-like end asymptotic to the intersection point  $z_\infty$ , the asymptotic operator is still  $f \mapsto -i\partial_t f - \varepsilon f$ , but now acting on functions  $f : [0, 1] \rightarrow \mathbb{C}$  with boundary conditions  $f(0) \in e^{i(c_0 + \varepsilon)}\mathbb{R}$  and  $f(1) \in e^{i c_0}\mathbb{R}$  for some real constant  $c_0$ . The eigenfunctions are  $ce^{(\pi n - \varepsilon)it + i(c_0 + \varepsilon)}$ ,  $c \in \mathbb{R}$ ,  $n \in \mathbb{Z}$ , with corresponding eigenvalues  $\pi n - 2\varepsilon$ . As before we say that an eigenfunction is *normalized* if  $|c| = 1$ .

**7.8.2. The asymptotic eigenfunction at an end.** Let  $\gamma \in \hat{\mathcal{O}}_{2g}$  and  $\gamma' \in \hat{\mathcal{O}}_{2g-l}$ . As before, the modifier  $*$  is placed as in  $\mathcal{M}_{J_-}^*(\gamma, \delta_0^l \gamma')$  to denote the subset of  $\mathcal{M}_{J_-}(\gamma_0, \delta_0^l \gamma')$  satisfying  $*$ . In particular,  $*$   $= (l_1, \dots, l_\lambda)$  means  $(l_1, \dots, l_\lambda)$  is a partition of  $l$  and we restrict to curves with  $\lambda$  ends at  $\delta_0$  with covering multiplicities  $l_1, \dots, l_\lambda$ .

We consider the asymptotics of  $\bar{u} \in \mathcal{M}_{J'}^{(l_1, \dots, l_\lambda)}(\gamma, \delta_0^l \gamma')$  near the negative end  $\delta_0^{l_j}$ . Let

$$\pi_{D_{\rho_0}} : \mathbb{R} \times (\mathbb{R}/2\mathbb{Z}) \times D_{\rho_0} \rightarrow D_{\rho_0}$$

be the projection to  $D_{\rho_0}$  and let

$$z_j = \pi_{D_{\rho_0}} \circ \bar{u} : (-\infty, s_0] \times (\mathbb{R}/2l_j\mathbb{Z}) \rightarrow D_{\rho_0}$$

be the projection of the negative end of  $\bar{u}$  which corresponds to  $\delta_0^{l_j}$ . The following asymptotic result is due to Hofer-Wysocki-Zehnder [HWZ1]:

**Lemma 7.8.3.** *There exist constants  $C_0, C_1 > 0$  such that the following holds: For any  $\bar{u} \in \mathcal{M}_{J'}^{(l_1, \dots, l_\lambda)}(\gamma, \delta_0^l \gamma')$  with a negative end at  $\delta_0^{l_j}$ , there exists an asymptotic eigenfunction  $f_j : \mathbb{R}/2l_j\mathbb{Z} \rightarrow \mathbb{C}$  given by  $f_j(t) = ce^{\pi i t/l_j}$  such that*

$$(7.8.2) \quad |z_j(s, t) - e^{(\pi/l_j - \varepsilon)s} f_j(t)| < C_0 e^{(\pi/l_j - \varepsilon + C_1)(s - s_0)}.$$

The function  $f_j(t)$  satisfying Equation (7.8.2) is called an *asymptotic eigenfunction for  $\bar{u}$  at the negative end  $\delta_0^{lj}$* .<sup>12</sup>

The following is due to Wendl [We4] and Hutchings-Taubes [HT2, Prop. 3.2]. (The proof of [HT2, Prop. 3.2] essentially proves the following lemma, but is not quite stated in the same way.)

**Lemma 7.8.4.** *There exists an arbitrarily small perturbation  $\tilde{J}'$  of  $\bar{J}'$  in  $(\bar{\mathcal{J}}')^{reg}$ , which is supported on  $\mathbb{R} \times \{\rho_1 < \rho < \rho_2\} \subset \mathbb{R} \times \bar{N}$  for  $0 < \rho_0 < \rho_1 < \rho_2$  arbitrarily small, such that the following holds:*

- ( $\star$ ) *for all  $\gamma$  and  $\gamma'$ , partitions  $(l_1, \dots, l_\lambda)$  of  $l$ ,  $j \in \{1, \dots, \lambda\}$  and  $r \in \mathbb{N} \cup \{0\}$ , the set of elements of  $\mathcal{M}_{\tilde{J}'}^{ind=r, (l_1, \dots, l_\lambda)}(\gamma, \delta_0^l \gamma') / \mathbb{R}$  with vanishing asymptotic eigenfunction at  $\delta_0^{lj}$  is a real codimension 2 submanifold.*

The real codimension 2 condition is due to the fact that  $\dim_{\mathbb{R}} E_{\pi/l_i + \varepsilon} = 2$ . Let  $(\bar{\mathcal{J}}')_{\star}^{reg} \subset (\bar{\mathcal{J}}')^{reg}$  be the (dense) subset of almost complex structures which satisfy ( $\star$ ).

**7.8.3. Definition of  $\tilde{U}_{m-1}$ .** Let  $\gamma \in \hat{\mathcal{O}}_{2g}$ ,  $\gamma' \in \hat{\mathcal{O}}_{2g-1}$ , and  $\tilde{J}' \in (\bar{\mathcal{J}}')_{\star}^{reg}$ . Let  $\bar{u} \in \mathcal{M}_{\tilde{J}'}^{I=ind=2, n^*=m-1}(\gamma, \delta_0 \gamma')$ . Since  $\tilde{J}'$  satisfies ( $\star$ ), the asymptotic eigenfunction of  $\bar{u}$  at  $\delta_0$  is nonzero. Hence we can associate a normalized asymptotic eigenfunction at  $\delta_0$  to  $\bar{u}$ .

Now let

$$\langle \tilde{U}_{m-1}(\gamma), \delta_0^p \gamma' \rangle = \begin{cases} 0, & \text{if } p \neq 1; \\ \# \left( \mathcal{M}_{\tilde{J}'}^{I=ind=2, n^*=m-1, f_{\delta_0}}(\gamma, \delta_0 \gamma') / \mathbb{R} \right), & \text{if } p = 1, \end{cases}$$

for a generic normalized asymptotic eigenfunction  $f_{\delta_0}$ . The modifier  $f_{\delta_0}$  stands for “the normalized asymptotic eigenfunction at the negative end  $\delta_0$  is  $f_{\delta_0}$ ”.

**7.8.4. The limit  $m \rightarrow \infty$ .** Suppose  $m \gg 0$ . Let  $\bar{h}_m : \bar{S} \xrightarrow{\sim} \bar{S}$  be a smooth extension of  $h : S \xrightarrow{\sim} S$  via  $\nu_m$ , as defined in Section 5.1.1, and let  $\bar{h}_\infty : \bar{S} \xrightarrow{\sim} \bar{S}$  be the smooth extension of  $h$  via  $\nu_\infty$ . The Hamiltonian structure on the suspension of  $\bar{h}_\infty$  will be written as  $(\alpha_{0,\infty}, \omega)$ . We then define  $\bar{\mathcal{J}}'_\infty$  as in Definition 5.3.9, with  $\alpha_0$  replaced by  $\alpha_{0,\infty}$ . Although the orbit  $\delta_0$  is degenerate, we still define the set  $\bar{\mathcal{P}} = \hat{\mathcal{P}} \cup \{\delta_0\}$  of orbits, the set  $\hat{\mathcal{O}}_k$  (resp.  $\bar{\mathcal{O}}_k$ ) of orbit sets constructed from  $\hat{\mathcal{P}}$  (resp.  $\bar{\mathcal{P}}$ ) which intersect  $\bar{S} \times \{0\}$  exactly  $k$  times, as in Section 5.3.2.

Let  $\bar{J}'_\infty \in \bar{\mathcal{J}}'_\infty$ . The moduli spaces  $\mathcal{M}_{\bar{J}'_\infty}^s(\gamma, \delta_0^l \gamma')$  are defined as follows: If  $l = 0$ , then the definition is the same as before. If  $l > 0$ , then we impose an exponential weight  $e^{-\varepsilon' s}$  at each negative end which approaches a multiple of  $\delta_0$ , where  $\varepsilon'$  is a small positive constant.

An almost complex structure  $\bar{J}'_\infty \in \bar{\mathcal{J}}'_\infty$  is *regular* if  $\mathcal{M}_{\bar{J}'_\infty}^s(\gamma, \delta_0^l \gamma')$  is transversely cut out for all  $0 < k \leq 2g$ ,  $l \geq 0$ ,  $\gamma \in \hat{\mathcal{O}}_k$ , and  $\gamma' \in \hat{\mathcal{O}}_{k-l}$ .

**Lemma 7.8.5.** *A generic  $\bar{J}'_\infty \in \bar{\mathcal{J}}'_\infty$  is regular and satisfies ( $\star$ ).*

<sup>12</sup>This is the terminology from [HT2], which is slightly different from that of the Hofer school.

*Proof.* The Fredholm theory for holomorphic curves with Morse-Bott asymptotics uses Sobolev spaces with exponential weights. The regularity of simply-covered moduli spaces in this setting is treated in Wendl [We4].  $\square$

As before, we write  $(\overline{\mathcal{J}}'_\infty)^{reg} \subset \overline{\mathcal{J}}'_\infty$  for the dense subset of regular almost complex structures and  $(\overline{\mathcal{J}}'_\infty)_\star^{reg} \subset (\overline{\mathcal{J}}'_\infty)^{reg}$  for the dense subset of almost complex structures which satisfy  $(\star)$ . Note that, if  $\overline{\mathcal{J}}'_\infty \in (\overline{\mathcal{J}}'_\infty)^{reg}$ , then nearby almost complex structures  $\overline{\mathcal{J}}' \in \overline{\mathcal{J}}'$  for  $m \gg 0$  are in  $(\overline{\mathcal{J}}')_\star^{reg}$ .

**Lemma 7.8.6.** *A sequence of curves  $\overline{u}_m \in \mathcal{M}_{\overline{\mathcal{J}}'_m}^{I=1}(\gamma, \delta_0^l \gamma')$  with  $m \rightarrow \infty$  converges (up to extracting subsequences) to a curve in  $\mathcal{M}_{\overline{\mathcal{J}}'_\infty}^{I=1}(\gamma, \delta_0^l \gamma')$ .*

*Proof.* The stable Hamiltonian vector field around  $\delta_0$  for  $m \gg 0$  is obtained by perturbing the (locally) Morse-Bott Hamiltonian vector field for  $m = \infty$  by a Hamiltonian function  $f$  on  $D^2$  which has a local maximum at  $z_\infty$ ; cf. [Bo2, Lemma 2.3].

Any sequence  $\overline{u}_m \in \mathcal{M}_{\overline{\mathcal{J}}'_m}^{I=1}(\gamma, \delta_0^l \gamma')$  has a subsequence which converges to a generalized Morse-Bott building by the Morse-Bott compactness theorem [Bo2, Theorem 4.16]. The limit must be negatively asymptotic to  $\delta_0^l \gamma'$  because there is no negative gradient trajectory of  $f$  on  $D^2$  that goes to  $z_\infty$ . Moreover the limit can have only one nontrivial level, because any further level must have  $I = 0$ .  $\square$

Let  $\overline{\mathcal{J}}'_\infty \in (\overline{\mathcal{J}}'_\infty)_\star^{reg}$ . By the compactness and regularity of moduli spaces, there are only finitely many curves  $\mathcal{C}_1, \dots, \mathcal{C}_r$ , modulo  $\mathbb{R}$ -translation, such that

$$\mathcal{C}_i \in \mathcal{M}_{\overline{\mathcal{J}}'_\infty}^{I=1, \tilde{n}=0, (l_{i1}, \dots, l_{i\lambda_i})}(\gamma_i, \delta_0^{l_i} \gamma'_i)$$

for some orbit sets  $\gamma_i, \gamma'_i \in \widehat{\mathcal{O}}_*$  and partition  $(l_{i1}, \dots, l_{i\lambda_i})$  of  $l_i$ . Here if  $\overline{u}$  is a curve with an end at  $\delta_0^p$ , then its ECH index is computed using the Conley-Zehnder index of  $\delta_0^p$  with respect to  $\overline{h}_m$ ,  $m \gg 0$ , and  $\tilde{n}(\overline{u})$  is defined as the intersection number of  $\overline{u}$  and the section at infinity. Let

$$f_{ij} : \mathbb{R}/2l_{ij}\mathbb{Z} \rightarrow \mathbb{C}, \quad t \mapsto c_{ij} e^{\pi i t / l_{ij}},$$

be the asymptotic eigenfunction corresponding the end  $\delta_0^{l_{ij}}$  of  $\mathcal{C}_i$ . The condition  $c_{ij} \neq 0$  follows from Lemma 7.8.5. We may therefore assume without loss of generality that all the  $f_{ij}$  are normalized.

### 7.8.5. Radial rays.

**Definition 7.8.7.** A *bad radial ray* is a radial ray  $\mathcal{R}_{\phi_0} = \{\phi = \phi_0, \rho \geq 0\}$  in  $\mathbb{C}$  which passes through

$$\{f_{ij}(t) \mid i = 1, \dots, r; j = 1, \dots, \lambda_i; 0 < t < 2l_{ij}; t \equiv 3/2 \pmod{2}\}.$$

A radial ray  $\mathcal{R}_{\phi_0}$  which is not bad is said to be *good*.

A good radial ray must exist and, as it is explained in the following remark, we can assume it is  $\mathcal{R}_\pi$ .

*Remark 7.8.8.* The set of bad radial rays is determined by  $(\mathbb{R} \times \overline{N}, \overline{J}'_\infty)$ . Strictly speaking, we should choose the set of endpoints  $E \subset \partial D^2$  as in Section 5.2.2 such that  $\mathcal{R}_{\phi_0+\pi}$  is a good radial ray and  $\phi_0 < \phi(y_j(m)) < \phi_0 + c(m)$  where  $c(m) \rightarrow 0$  as  $m \rightarrow \infty$ . After a rotational coordinate change of  $D^2$ , we may assume that  $\mathcal{R}_\pi$  is a good radial ray and  $0 < \phi(y_j(m)) < c(m)$ .

**7.9. The rescaled function.** In this subsection we will be using limiting arguments in which  $m \rightarrow \infty$  and  $\overline{h}_m \rightarrow \overline{h}_\infty$ ; see Section 7.8.4. Hence many of the almost complex structures and moduli spaces will have an additional subscript  $m$ , where  $m = \infty$  is also a possibility. For example,  $\overline{J}'_m$  and  $\overline{J}_{-,m}$  refer to  $\overline{J}'$  and  $\overline{J}_-$  with respect to  $m$ . Let  $\overline{J}'_\infty \in (\overline{J}'_\infty)_\star^{reg}$  and let  $\overline{J}'_m \in (\overline{J}'_m)_\star^{reg}$  be a nearby almost complex structure with respect to the integer  $m \gg 0$ . Let  $\overline{J}_{-,m} \in \overline{J}_{-,m}^{reg}$  be an almost complex structure which restricts to  $\overline{J}'_m$  and let  $\overline{J}_{-,m}^\diamond$  be  $(\varepsilon, U)$ -close to  $\overline{J}_{-,m}$ .

We write  $\pi : V \rightarrow D_{\rho_0}$  for the projection which was denoted  $\pi_{D_{\rho_0}}$  in previous sections.

Let  $m_i$  and  $\overline{u}_{ij}$  be sequences satisfying the following properties:

- (S1)  $\lim_{i \rightarrow \infty} m_i = \infty$ ;
- (S2)  $\overline{u}_{ij} \in \mathcal{M}_{\overline{J}_{-,m_i}^\diamond}^{I=3, n^*=m_i}(\gamma, \mathbf{y}; \overline{\mathbf{m}})$ , where  $\mathbf{y} = \{x_l\} \cup \mathbf{y}'$  or  $\{x'_l\} \cup \mathbf{y}'$  for some  $l$ ; and
- (S3) for all  $i \in \mathbb{N}$  and  $\kappa, \nu > 0$ , there exists  $j_{i,\kappa,\nu}$  such that, if  $j \geq j_{i,\kappa,\nu}$ , then  $\overline{u}_{ij}$  is  $(\kappa, \nu)$ -close (cf. Definition 6.5.2) to a  $\overline{J}_{-,m_i}^\diamond$ -holomorphic building  $\overline{u}_{i\infty} = \cup_l \overline{v}_{l,i}$  with  $\overline{v}'_{0,i} \neq \emptyset$ .

Moreover, assume that one of the following holds:

- (S4') the first negative end at  $\delta_0$  in the building  $\overline{u}_{i\infty}$  has multiplicity one, or
- (S4'') the first negative end at  $\delta_0$  in the building  $\overline{u}_{i\infty}$  has multiplicity two.

Property (S3) is a consequence of the fact that the sequence  $\overline{u}_{ij}$  converges in the SFT sense to the building  $\overline{u}_{i\infty}$  for each fixed  $i$ . Property (S4') corresponds to Cases (1) and (2) in Theorem 7.7.1 and Property (S4'') corresponds to the remaining cases.

### 7.9.1. Asymptotics.

**Lemma 7.9.1.** *Let  $\overline{v}$  be a  $\overline{J}'_m$ -holomorphic curve in  $\mathbb{R} \times \overline{N}$  with ECH index  $I \leq 2$  for  $\overline{J}'_m \in (\overline{J}'_m)_\star^{reg}$  and let  $\tilde{v} : (-\infty, s_0] \rightarrow \mathbb{R} \times \overline{N}$  be a negative end of  $\overline{v}$  which is asymptotic to a multiple cover of  $\delta_0$ . Then for every  $\varepsilon > 0$  there exist positive constants  $\overline{R}$  and  $\kappa', \kappa < \varepsilon$  such that  $\frac{\kappa'}{\kappa} > \frac{1}{\varepsilon}$  and*

$$|\pi \circ \tilde{v}(-\overline{R}, t) - \kappa' f(t)| \leq \frac{\kappa}{2},$$

for all  $t$ , where  $f$  is the normalized asymptotic eigenfunction for  $\tilde{v}$ .

*Proof.* This is a rephrasing of Lemma 7.8.3, in view of Lemma 7.8.4. □

**7.9.2. Truncation.** Let  $F$  be a Riemann surface and  $\bar{u} : F \rightarrow (\bar{W}_-, \bar{J}_-)$  a holomorphic map. We denote by  $p : F \rightarrow B_-$  the map  $\bar{\pi}_{B_-} \circ \bar{u}$  and by  $s : F \rightarrow \mathbb{R}$ ,  $t : F \rightarrow S^1$  the functions obtained by composing  $p$  with the coordinates  $(s, t)$  on  $B_-$ . The functions  $(s, t)$  give local coordinates on  $F$  outside the critical points of  $p$ .

Let  $\bar{g}$  be the restriction of an  $s$ -invariant Riemannian metric on  $\bar{W}'$  to  $\bar{W}_-$  and let  $d$  be the topological metric induced by  $\bar{g}$ .

**Definition 7.9.2** (Truncation). A *truncation*  $(\tilde{u}, \tilde{F}, \{R^{(l)}\}_{l=1}^a, \tilde{\mathbf{m}}, \mathfrak{e}^\pm, \kappa')$  of a holomorphic map  $\bar{u} : F \rightarrow (\bar{W}_-, \bar{J}_-)$  is a tuple where:

- $\tilde{F}$  is a subsurface of  $F$  and  $\tilde{u}$  is the restriction of  $\bar{u}$  to  $\tilde{F}$ ;
- the restriction of  $p$  to each  $\tilde{F} \cap \{R^{(l)} \leq s \leq R^{(l+1)}\}$  is a branched cover;
- $\sup_{x \in \tilde{u}(\tilde{F})} d(x, \sigma_\infty^-) \leq 2\kappa'$ ; if  $d(\bar{u}(y), \sigma_\infty^-) \leq \frac{\kappa'}{2}$ , then  $y \in \tilde{F}$ ;
- $\tilde{\mathbf{m}} \in \tilde{F}$  is the unique point such that  $\tilde{u}(\tilde{\mathbf{m}}) = (\bar{\mathbf{m}}_-)$ ;
- $\mathfrak{e}^+$  (resp.  $\mathfrak{e}^-$ ) is the union of components of  $\partial\tilde{F} - p^{-1}(\partial B_-)$  for which  $ds(\mathbf{n}) > 0$  (resp.  $< 0$ ), where  $\mathbf{n}$  is the outward normal vector field along  $\partial\tilde{F}$ .

**Definition 7.9.3.** A *good truncation*

$$(\tilde{u}, \tilde{F}, \{R^{(l)}\}_{l=1}^a, \tilde{\mathbf{m}}, \mathfrak{e}^\pm, \kappa, \kappa')$$

is a truncation  $(\tilde{u}, \tilde{F}, \{R^{(l)}\}_{l=1}^a, \tilde{\mathbf{m}}, \mathfrak{e}^\pm, \kappa')$ , together with a constant  $\kappa > 0$ , such that  $\frac{\kappa'}{\kappa} > 2$  and:

- (G)  $|\pi \circ \tilde{u}|_{\mathfrak{e}^\pm} - f| \leq \kappa$ , where  $f$  is an asymptotic eigenfunction of  $\delta_0$  or  $z_\infty$  (as appropriate) on each component of  $\mathfrak{e}^\pm$  and  $|f| \geq \kappa'$ .

**Lemma 7.9.4.** Let  $m_i$  and  $\bar{u}_{ij}$  be sequences satisfying (S1)–(S3) and either (S4') or (S4''). Then there is a sequence  $j(i)$  such that the following hold for all  $j \geq j(i)$ :

- (1)  $\bar{u}_{ij}$  admits a good truncation

$$(\tilde{u}_{ij}, \tilde{F}_{ij}, \{R_{ij}^{(l)}\}_{l=1}^a, \tilde{\mathbf{m}}_{ij}, \mathfrak{e}_{ij}^\pm, \kappa_i, \kappa'_i);$$

- (2)  $\lim_{i \rightarrow \infty} \kappa'_i = \lim_{i \rightarrow \infty} \kappa_i = 0$  and  $\lim_{i \rightarrow \infty} \frac{\kappa'_i}{\kappa_i} = +\infty$ ;

*Proof.* We prove the lemma in Case (S4'), where the notation is simpler, i.e., we can use  $(s, t)$  as global coordinates on  $\tilde{F}_{ij}$ . In this case  $a = 2$ . Case (S4'') is conceptually the same.

Fix a sequence  $\varepsilon_i \rightarrow 0$ . Then, by Lemma 7.9.1, for each  $i$  there exist  $\kappa'_i$ ,  $\kappa_i$  and  $\bar{R}_i$  which satisfy:

- $\kappa'_i, \kappa_i < \varepsilon_i$  and  $\frac{\kappa'_i}{\kappa_i} > \frac{1}{\varepsilon_i}$ ; and
- $|\pi \circ \tilde{u}_i(-\bar{R}_i, t) - \kappa'_i f_i(t)| \leq \frac{\kappa_i}{2}$  for all  $t$ , where  $f_i$  is the normalized asymptotic eigenfunction for a negative end  $\tilde{v}_i$  of a  $\bar{J}'_{m_i}$ -holomorphic curve  $\bar{v}_i$  of ECH index  $I \leq 2$  that limits to  $\delta_0$ .

On the other hand, by (S3) and Definition 6.5.2, for all  $i$  there exists  $j(i)$  such that if  $j > j(i)$  then there exists  $R'_{ij}$  such that

$$|\pi \circ \tilde{u}_{ij}(R'_{ij} - \bar{R}_i, t) - \pi \circ \tilde{v}_i(-\bar{R}_i, t)| \leq \frac{\kappa_i}{2}$$

for all  $t$ . Setting  $R_{ij}^{(1)} = R'_{ij(i)} - \bar{R}_i$ , we obtain:

$$\begin{aligned} \left| \pi \circ \tilde{u}_{ij}(R_{ij}^{(1)}, t) - \kappa'_i f_i(t) \right| &\leq \left| \pi \circ \tilde{u}_{ij}(R_{ij}^{(1)}, t) - \pi \circ \tilde{v}_i(-\bar{R}_i, t) \right| \\ &\quad + \left| \pi \circ \tilde{v}_i(-\bar{R}_i, t) - \kappa'_i f_i(t) \right| \leq \kappa_i. \end{aligned}$$

Similar considerations also hold at the negative truncated end.  $\square$

*Remark 7.9.5.* As  $i \rightarrow \infty$ , the curves  $\bar{v}_i$  approach one of the  $\mathcal{C}_a$  from Section 7.8.4, modulo  $\mathbb{R}$ -translation, by Lemma 7.8.6. Hence the normalized asymptotic eigenfunctions  $f_i$  limit to the normalized eigenfunction  $f_{ab}$  for  $\mathcal{C}_a$  at the negative end  $\delta_0$  as  $i \rightarrow \infty$ .

**7.9.3. Ansatz.** We define  $\tilde{z}_i = \pi \circ \tilde{u}_{ij(i)}$  and, to simplify computations, we will make the ansatz

$$(7.9.1) \quad \tilde{z}_i = e^{-\varepsilon_i s} \tilde{w}_i,$$

where  $\varepsilon_i = \frac{\pi}{m_i}$ .

**Lemma 7.9.6.** *The functions  $\tilde{w}_i : \tilde{F}_i \rightarrow \mathbb{C}$  defined by Equation (7.9.1) are holomorphic with respect to the standard complex structure on  $\mathbb{C}$ .*

*Proof.* We give the proof for Case (S4''); Case (S4') is similar but simpler. The functions  $(s, t)$  give local conformal coordinates on  $\tilde{F}_i$  outside the branch locus because  $p_i : \tilde{F}_i \rightarrow B_-$  is holomorphic. Then  $\tilde{z}_i : \tilde{F}_i \rightarrow \mathbb{C}$  satisfies Equation (7.8.1):

$$\partial_s \tilde{z}_i + i \partial_t \tilde{z}_i + \varepsilon_i \tilde{z}_i = 0.$$

If we plug in Equation (7.9.1) into Equation (7.8.1) we obtain:

$$-\varepsilon_i e^{-\varepsilon_i s} \tilde{w}_i + e^{-\varepsilon_i s} \partial_s \tilde{w}_i + i e^{-\varepsilon_i s} \partial_t \tilde{w}_i + \varepsilon_i e^{-\varepsilon_i s} \tilde{w}_i = 0.$$

Hence  $\partial_s \tilde{w} + i \partial_t \tilde{w}_i = 0$ , so  $\tilde{w}_i$  is a holomorphic map to the standard complex line  $(\mathbb{C}, i)$  in the complement of the branch point. Then it is holomorphic everywhere by the removal of singularities.  $\square$

**7.9.4. Rescaling.** Consider the diagonal subsequence  $\bar{u}_i = \bar{u}_{ij(i)}$ ,  $i = 1, 2, \dots$ . We abbreviate  $\tilde{F}_i = \tilde{F}_{ij(i)}$ ,  $p_i = p_{ij(i)}|_{\tilde{F}_{ij(i)}}$ ,  $\tilde{m}_i = \tilde{m}_{ij(i)}$ , etc. Fix  $R'_0, R_0 > 0$  and let  $K_i$  be the connected component of  $p_i^{-1}(B_- \cap \{-R'_0 \leq s \leq R_0\})$  containing  $\tilde{m}$ . By passing to a subsequence we may assume that the topology of  $K_i$  and  $\tilde{F}_i - K_i$  are constant. In order to simplify the exposition, we will assume that this is the case for all  $i$ .

**Definition 7.9.7.** We define  $C_i = \sup_{z \in K_i} |\tilde{w}_i(z)|$  and  $w_i = \tilde{w}_i / C_i$ .

*Properties of  $w_i$ .* The holomorphic maps  $w_i : \tilde{F}_i \rightarrow \mathbb{C}$  satisfy the following:

- (1)  $\sup_{z \in K_i} |w_i(z)| = 1$ ;
- (2) there is a unique point  $\tilde{m}_i \in \tilde{F}_i$  such that  $w_i(\tilde{m}_i) = 0$ ; the zero  $\tilde{m}_i$  is a simple zero;
- (3) if  $x \in \partial \tilde{F}_i$  and  $p_i(x) = (s, t) \in \partial B_-$ , then  $w_i(x) \in e^{i(\varepsilon_i t + \phi_i)} \mathbb{R}^+$  with  $\varepsilon_i = \frac{\pi}{m_i}$  and  $\lim_{i \rightarrow \infty} \phi_i = 0$  for some  $\phi_i$  depending on the boundary component of  $\partial \tilde{F}_i$  containing  $x$ ;
- (4) at the truncated ends the inequality

$$(7.9.2) \quad |w_i|_{\mathfrak{e}_i^\pm} - f_i| \leq \tilde{\kappa}_i$$

holds, where  $f_i$  is a (not necessarily normalized) asymptotic eigenfunction of  $\delta$  or  $z_\infty$  on each component of  $\mathfrak{e}_i^\pm$ ,  $|f_i| \geq \tilde{\kappa}'_i$ , and  $\tilde{\kappa}_i$  and  $\tilde{\kappa}'_i$  satisfy  $\tilde{\kappa}'_i/\tilde{\kappa}_i \rightarrow \infty$  as  $i \rightarrow \infty$ .

*Remark 7.9.8.* The loops of  $w_i|_{\mathfrak{e}_i^\pm}$  have winding number 1 around the origin when  $i$  is sufficiently large: in fact  $\tilde{\kappa}'_i > \tilde{\kappa}_i$ , so the linear homotopy between  $w_i|_{\mathfrak{e}_i^\pm}$  and  $f_i(t)$  is contained in  $\mathbb{C}^\times$ .

**Lemma 7.9.9.** *After passing to a subsequence,  $w_i|_{K_i}$  converges to a nonconstant function  $w_\infty|_{K_0}$ .*

*Proof.* After passing to a subsequence we may assume that  $K_i$  converges to a Riemann surface  $K_0$  with boundary. Since  $w_i|_{K_i}$  is uniformly bounded, the lemma follows from Montel's theorem.  $\square$

From now on we assume that we have passed to a subsequence so that Lemma 7.9.9 holds. In the following subsections we analyze the convergence of  $w_i$ , not just on the uniformly bounded part  $K_i$ . This involves techniques of SFT compactness.

**7.9.5. Energy bound.** Following Hofer [Ho1], we define an *energy* for holomorphic functions on Riemann surfaces with boundary and punctures.

**Definition 7.9.10** (Energy). Let  $\mathcal{C}$  be the set of smooth functions  $\varphi : [0, \infty) \rightarrow [0, 1]$  such that  $\varphi(0) = 0$  and  $\varphi'(r) \geq 0$  for all  $r \in [0, \infty)$ . If  $F$  is a Riemann surface and  $u : F \rightarrow \mathbb{C}$  a holomorphic function, we define the *energy* of  $u$  as

$$E(u) = \sup_{\varphi \in \mathcal{C}} \int_F u^* d(\varphi(r) d\theta),$$

where  $(r, \theta)$  are polar coordinates on  $\mathbb{C}$ .

*Remark 7.9.11.* If we identify  $\mathbb{C}^\times \xrightarrow{\sim} \mathbb{R} \times S^1$  using the log map, then

$$\sup_{\varphi \in \mathcal{C}} \int_{F - u^{-1}(0)} u^* d(\varphi(r) d\theta)$$

agrees with the usual expression for the Hofer energy of  $\log \circ u : F - u^{-1}(0) \rightarrow \mathbb{R} \times S^1$ .

**Lemma 7.9.12.** *The sequence  $w_i$  has uniformly bounded energy.*

*Proof.* By Stokes' theorem,

$$\int_{\tilde{F}_i} w_i^*(d\phi(r)d\theta) = \int_{\partial\tilde{F}_i} w_i^*(\phi(r)d\theta).$$

The boundary  $\partial\tilde{F}_i$  is the union of  $\mathfrak{e}_i^\pm$ ,  $\mathfrak{f}_{ik} = p_i^{-1}(\{t = k\}) \cap \partial\tilde{F}_i$ ,  $k = 0, 1$ , and  $\mathfrak{f}_{i2} = p_i^{-1}(\{1 < t < 2\}) \cap \partial\tilde{F}_i$ , all oriented using the boundary orientation. Let us write  $\eta = w_i^*(\phi(r)d\theta)$ . If  $\mathfrak{c}$  is a component of  $\mathfrak{e}_i^\pm$ , then  $\int_{\mathfrak{c}} \eta \approx 2\pi \deg(w_i|_{\mathfrak{c}})$  if  $\mathfrak{c}$  is a circle and  $\int_{\mathfrak{c}} \eta \approx \frac{\pi}{m}$  if  $\mathfrak{c}$  is an arc. The argument works because we may take  $w_i|_{\mathfrak{c}}$  to be  $C^l$ -close (not just  $C^0$ -close) to an asymptotic eigenfunction, with  $l \geq 1$ . This follows from the exponential decay estimates of [HWZ1]; also see [HT2, Lemma 2.3]. Moreover, for  $k = 0, 1$ ,  $\int_{\mathfrak{f}_{ik}} \eta = 0$  since  $w|_{\mathfrak{f}_{ik}}$ ,  $k = 0, 1$ , projects to a radial ray. Finally,  $\int_{\mathfrak{f}_{i2}} \eta < 0$  since  $w|_{\mathfrak{f}_{i2}}$  always has a component in the negative  $\theta$ -direction; here it is important to remember that we are projecting using balanced coordinates. This proves the lemma.  $\square$

*Remark 7.9.13.* Observe that  $E(cu) = E(u)$  where  $c \in \mathbb{C}^\times$ .

**7.9.6. Bubbling.** The goal of this subsection is to eliminate certain types of bubbling.

Let  $(\tilde{F}^i, \tilde{g}_i)$ ,  $i \in \mathbb{N}$ , be a sequence of Riemannian manifolds which are compatible with the complex structures and have injectivity radii which are uniformly bounded below. For example, one could obtain the metrics  $\tilde{g}_i$  by scaling the compatible hyperbolic metrics by a conformal factor so that the thick parts remain hyperbolic, while the thin parts become flat cylinders. We will use these metrics to compute the pointwise norm of the differential of the functions  $w_i$  which are denoted by  $|w'_i|$ .

**Lemma 7.9.14.** *For every compact set  $K \subset B_-$ , there is a constant  $C_K > 0$  such that  $|w_i(z)| < C_K$  and  $|w'_i(z)| < C_K$  for all  $z \in p_i^{-1}(K)$  and all  $i \in \mathbb{N}$ .*

*Proof.* The lemma holds for  $K \cap K_i$  by Lemma 7.9.9. Hence it suffices to consider  $K \cap K_i^c$ , where  $K_i^c := \tilde{F}_i - \text{int}(F_i)$ . Since  $w_i(K_i^c) \subset \mathbb{C}^\times$ , we compose with  $\log : \mathbb{C}^\times \rightarrow \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$  and consider  $\zeta_i = \log \circ w_i|_{K_i^c}$ . The sequence  $\zeta_i$  has uniformly bounded energy by Lemma 7.9.12. If  $|\zeta'_i|$ ,  $i \in \mathbb{N}$ , is not uniformly bounded on  $K_i^c$ , then the usual bubbling analysis yields a nonconstant finite energy plane  $\tilde{v}_\infty^+ : \mathbb{C} \rightarrow \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$  or half-plane  $\tilde{v}_\infty^+ : \mathbb{H} \rightarrow \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ ; see the proof of [Ho1, Theorem 31]. If  $\tilde{v}_\infty^+$  is a half-plane, then  $\tilde{v}_\infty^+(\partial\mathbb{H})$  is contained in a line  $\mathbb{R} \times \{pt\} \subset \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$  as a consequence of the boundary conditions for the maps  $w_i$ , and we can double  $\tilde{v}_\infty^+$  to obtain a nonconstant finite energy plane by the Schwarz reflection principle. On the other hand, by [Ho1, Lemma 28], there are no nonconstant finite energy planes in  $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ , a contradiction. Hence  $|\zeta'_i|$ ,  $i \in \mathbb{N}$ , is uniformly bounded on  $K_i^c$ . This in turn implies uniform bounds on  $|w_i|$  and  $|w'_i|$  on  $K \cap (K_i^c)$ .  $\square$

**7.9.7. Case (S4').** In Cases (S4') and (S4'') we often pass to a subsequence without explicit mention.



Suppose we are in Case (S4'). Recall the compactification  $cl(B_-)$  of  $B_-$  defined in Section 5.1.1, obtained by adjoining the points at infinity  $p_+, p_-$ . Here  $cl(B_-)$  is isomorphic to the closed unit disk and  $p_+$  is a marked point in the interior and  $p_-$  is a marked point on the boundary under this identification.

**Theorem 7.9.15.** *The sequence  $(w_i)_{i \in \mathbb{N}}$  converges uniformly on compact subsets to a holomorphic map  $w_\infty : B_- \rightarrow \mathbb{C}$  such that:*

- (1)  $w_\infty(\partial B_-) \subset \mathbb{R}^+$  and  $w_\infty(\overline{m}^b) = 0$ ;
- (2)  $\lim_{s \rightarrow +\infty} |w_\infty(s, t)| = +\infty$  and  $\lim_{s \rightarrow +\infty} \frac{w_\infty(s, t)}{|w_\infty(s, t)|} = f_{ab}(t)$ ;
- (3)  $\lim_{s \rightarrow -\infty} w_\infty(s, t) = c \in \mathbb{R}^+$ ;
- (4)  $w_\infty|_{int(B_-)}$  is biholomorphism onto its image. In particular  $\overline{m}^b$  is the unique zero of  $w_\infty|_{cl(B_-)}$  and is simple;
- (5)  $w_\infty$  extends to a holomorphic map  $cl(B_-) \rightarrow \mathbb{CP}^1$ , still called  $w_\infty$ , such that  $w_\infty(\partial cl(B_-)) = [a_1, a_2] \subset \mathbb{R}^+$  and  $w_\infty(p_+) = \infty$ .

*Proof.* The uniform convergence on compact subsets is a consequence of the bounds from Lemma 7.9.14. (1) is immediate from the convergence, together with the observation that the angles between the arcs in  $\overline{\mathfrak{a}}$  tend to 0 as  $m_i \rightarrow \infty$ .

(2) Let  $K_i^+$  (resp.  $K_i^-$ ) be the component of  $\tilde{F}_i - int(K_i)$  with positive (resp. negative)  $s$ -coordinates. We expand  $w_i$  and  $w_\infty$  in Fourier series on  $K_i^+$ :

$$w_i(s, t) = \sum_{n=-\infty}^{+\infty} a_n^i e^{\pi n(s+it)}, \quad w_\infty(s, t) = \sum_{n=-\infty}^{+\infty} a_n^\infty e^{\pi n(s+it)}.$$

By the uniform convergence of  $w_i$  to  $w_\infty$ ,  $\lim_{i \rightarrow \infty} a_n^i = a_n^\infty$  for all  $n$ . Then (2) is equivalent to the conditions:

- (i)  $\lim_{i \rightarrow \infty} \frac{a_1^i}{|a_1^i|} e^{i\pi t} = f_{ab}(t)$ ; and
- (ii)  $\lim_{i \rightarrow \infty} a_n^i = 0$  when  $n \geq 2$ .

Since we are in Case (S4'),  $\{R_i^{(l)}\}_{l=1}^a = \{R_i^{(1)}, R_i^{(2)}\}$ , i.e.,  $a = 2$ . By Equation (7.9.2),

$$\left| \frac{1}{2} \int_0^2 \left( w_i(R_i^{(2)}, t) - f_i(t) \right) e^{-\pi i n t} dt \right| \leq \tilde{\kappa}_i,$$

for all  $n \in \mathbb{Z}$ . On the other hand, if we write  $f_i(t) = c_i e^{\pi i t}$ , where  $f_i(t)$  is not necessarily normalized, then

$$\frac{1}{2} \int_0^2 \left( w_i(R_i^{(2)}, t) - f_i(t) \right) e^{-\pi i n t} dt = a_n^i e^{\pi n R_i^{(2)}} - c_i \delta_{in},$$

where  $\delta_{in}$  is the Kronecker delta. Hence

- (a)  $|a_n^i| \cdot e^{\pi n R_i^{(2)}} \leq \tilde{\kappa}_i$  for all  $n \neq 1$ ; and
- (b)  $|a_1^i e^{\pi R_i^{(2)}} - c_i| \leq \tilde{\kappa}_i$ .

We prove (ii). Arguing by contradiction, suppose that  $\lim_{i \rightarrow \infty} |a_n^i| \neq 0$  for some  $n \geq 2$ . By (a), there exists  $C > 0$  such that  $Ce^{2\pi R_i^{(2)}} < \tilde{\kappa}_i$  for all  $i$ . Since  $R_i^{(2)} \rightarrow \infty$ , this implies that  $|a_1^i e^{\pi R_i^{(2)}}| < \tilde{\kappa}_i$  for all  $i$ . On the other hand,  $|c_i| \geq \tilde{\kappa}_i'$  and  $\lim_{i \rightarrow \infty} \frac{\tilde{\kappa}_i'}{\tilde{\kappa}_i} = +\infty$ , which contradicts (b).

Next we prove (i). We claim that  $\lim_{i \rightarrow \infty} a_1^i \neq 0$  for topological reasons. Indeed, if  $\lim_{i \rightarrow \infty} a_1^i = 0$ , then  $\lim_{i \rightarrow \infty} a_n^i = 0$  for all  $n \geq 1$  by (ii) and the curve  $w_i|_{s=R_0}$  has nonpositive winding number around 0 when  $i \gg 0$ . On the other hand,  $w_i|_{s=R_0}$  is homotopic to the curve  $w_i|_{s=R_i^{(2)}}$  in  $\mathbb{C}^\times$ , a contradiction. This proves the claim.

Finally,  $\lim_{i \rightarrow \infty} \left| a_1^i \cdot \frac{e^{\pi R_i^{(2)}}}{|c_i|} - \frac{c_i}{|c_i|} \right| = 0$ , since  $\lim_{i \rightarrow \infty} \frac{\tilde{\kappa}_i}{\tilde{\kappa}_i'} = \lim_{i \rightarrow \infty} \frac{\tilde{\kappa}_i}{|c_i|} = 0$ . Hence

$$\lim_{i \rightarrow \infty} \frac{a_1^n}{|a_1^n|} e^{\pi i t} = f_{ab}(t),$$

which proves (i).

(3) This is similar to (2). We expand  $w_i$  in Fourier series on  $K_i^-$ :

$$w_i(s, t) = \sum_{-\infty}^{+\infty} a_n^i e^{\varepsilon_i i} e^{(\pi n - \varepsilon_i)(s + it)}.$$

By the uniform convergence,  $\lim_{s \rightarrow -\infty} a_n^i = a_n^\infty$  and we can similarly prove that  $a_n^\infty = 0$  for all  $n < 0$ . This implies (3) because the normalized eigenfunctions converge to a constant as  $i \rightarrow +\infty$ .

(4) Since  $H_2(cl(B_-), \partial cl(B_-)) \cong H_2(\mathbb{CP}^1, [a_1, a_2]) \cong \mathbb{Z}$ , we have a well-defined notion of degree for  $w_\infty$ . Moreover, as in the closed case, the degree is equal to the cardinality of the inverse image of a regular value in  $\mathbb{CP}^1 - [a_1, a_2]$ . Hence  $\deg w_\infty|_{int(B_-)} = 1$  because  $w_\infty^{-1}(\infty) = \{0\}$  and 0 is a simple pole. This implies that  $w_\infty : int(B_-) \rightarrow \mathbb{C} - [a_1, a_2]$  is a biholomorphism.

(5) follows from (2) and (3).  $\square$

**7.9.8. Case (S4'').** We give a brisk treatment of the construction of the SFT limit, mostly pointing out the differences with (S4'). The main difference is that the projection  $p_i : \tilde{F}_i \rightarrow B_-$  is a branched double cover over its image with a single branch point  $b_i \in B_-$ , and we must analyze different cases depending on the behavior of the branch point as  $i \rightarrow \infty$ .

There are five cases:

- (a)  $\lim_{i \rightarrow \infty} b_i = b_\infty \in int(B_-)$ ,
- (b)  $\lim_{i \rightarrow \infty} b_i = b_\infty \in \partial B_-$ ,
- (c)  $\lim_{i \rightarrow \infty} s(b_i) = +\infty$ ,
- (d')  $\lim_{i \rightarrow \infty} s(b_i) = -\infty$  and  $d(b_i, \partial B_-) \geq C$ , where  $C > 0$  is a constant, or
- (d'')  $\lim_{i \rightarrow \infty} s(b_i) = -\infty$  and  $d(b_i, \partial B_-) \rightarrow 0$ .

The sequence  $p_i : \tilde{F}_i \rightarrow B_-$  converges in the SFT sense to a 1- or 2-level holomorphic building  $p_\infty = p_\infty^0, p_\infty^0 \cup p_\infty^1$ , or  $p_\infty^{-1} \cup p_\infty^0$ , where  $p_\infty^{-1}, p_\infty^0, p_\infty^1$  are branched covers of  $B, B_-, B'$ , respectively. We have a 1-level building in Cases (a) and (b) and a 2-level building in Cases (c), (d'), and (d''). Let  $p_\infty^m : \tilde{F}_\infty^m \rightarrow B_-$  be the restriction of  $p_\infty^0$  to the component containing the limit  $\tilde{m}_\infty$  of  $\tilde{m}_i$  and let  $p_\infty^b : \tilde{F}_\infty^b \rightarrow B$  or  $B'$  be the restriction of  $p_\infty^{-1}, p_\infty^0$ , or  $p_\infty^1$  to the component containing the limit  $b_\infty$  of  $b_i$ . If  $p_\infty^m = p_\infty^b$  (i.e., in Cases (a) and (b)) we drop the superscripts.

Near all the punctures of  $\tilde{F}_\infty^*$ ,  $\star \in \{m, b\}$ , we use cylindrical or rectangular coordinates  $(s, t)$  induced by the coordinates  $(s, t)$  on  $B_-$  by pullback, after a possible translation in the  $s$ -direction.

For  $\star \in \{\emptyset, ', -\}$ , we denote the compactification of  $B^*$ , obtained by adjoining the points at infinity  $p_\pm^*$ , by  $cl(B^*)$ . Similarly, let  $cl(\tilde{F}_\infty^*)$ ,  $\star \in \{m, b\}$ , be the compactification of  $\tilde{F}_\infty^*$ , obtained by adjoining  $q_{\pm, j}^*$ , where  $j \in \{1\}$  or  $\{1, 2\}$ , depending on the number of ends. (If there is only one end, we suppress the index.) The maps  $p_\infty^* : \tilde{F}_\infty^* \rightarrow B^*$ , can be compactified to  $p_\infty^* : cl(\tilde{F}_\infty^*) \rightarrow cl(B^*)$ , where  $q_{\pm, j}^*$  is mapped to  $p_\pm^*$ .

In Case (a),  $\tilde{F}_\infty$  is an annulus with a puncture  $q_+$  in the interior and a puncture  $q_{-, j}$  on each boundary component. In Case (b),  $\tilde{F}_\infty$  is a disk with one puncture  $q_+$  in the interior, two punctures  $q_{-, j}$ ,  $j = 1, 2$ , on the boundary, and two boundary points  $b_{\infty, j}$ ,  $j = 1, 2$ , which are glued together to give  $b_\infty$ . The points  $q_{-, j}$  and  $b_{\infty, j}$  alternate along the boundary. In Case (c),  $\tilde{F}_\infty^m$  is a disk with a puncture  $q_+^m$  in the interior and a puncture  $q_-^m$  on the boundary, and  $\tilde{F}_\infty^b$  is a sphere with three punctures  $q_+^b, q_-^b, q_{-, j}^b$ ,  $j = 1, 2$ . In Case (d'),  $\tilde{F}_\infty^m$  is a disk with a puncture  $q_+^m$  in the interior and two punctures  $q_{-, j}^m$  on the boundary and  $\tilde{F}_\infty^b$  is a disk with four punctures  $q_{\pm, j}^b$  on the boundary. In Case (d''),  $\tilde{F}_\infty^m$  is a disk with a puncture  $q_+^m$  in the interior and two punctures  $q_{-, j}^m$  on the boundary and  $\tilde{F}_\infty^b$  is a disk with four punctures  $q_{\pm, j}^b$  on the boundary and two boundary points  $b_{\infty, j}$ ,  $j = 1, 2$ , identified.

Cases (a) and (b) are similar to Case (S4'), while the situation in Cases (c), (d') and (d'') is complicated by the fact that the limit is a 2-level holomorphic building.

*Cases (a) and (b).*

**Theorem 7.9.16.** *Suppose  $\lim_{i \rightarrow \infty} b_i = b_\infty \in B_-$ . Then  $(w_i)_{i \in \mathbb{N}}$  converges to a holomorphic map  $w_\infty : \tilde{F}_\infty \rightarrow \mathbb{C}$  such that:*

- (1)  $w_\infty(\partial \tilde{F}_\infty) \subset \mathbb{R}^+$ ;
- (2) *at the positive puncture  $q_+$ ,*

$$\lim_{s \rightarrow +\infty} |w_\infty(s, t)| = +\infty, \quad \lim_{s \rightarrow +\infty} \frac{w_\infty(s, t)}{|w_\infty(s, t)|} = f_{ab}(t);$$

- (3) *at the negative punctures  $q_{-, j}$ ,  $j = 1, 2$ ,*  $\lim_{s \rightarrow -\infty} w_\infty(s, t) = c_j \in \mathbb{R}^+$ ;

- (4a) in Case (a),  $w_\infty$  extends to a holomorphic map  $w_\infty : cl(\tilde{F}_\infty) \rightarrow \mathbb{CP}^1$  such that  $w_\infty : int(cl(\tilde{F}_\infty)) \rightarrow \mathbb{CP}^1 - ([a_1, a_2] \sqcup [a_3, a_4])$  is a biholomorphism;  
 (4b) in Case (b),  $w_\infty$  extends to a holomorphic map  $w_\infty : cl(\tilde{F}_\infty) \rightarrow \mathbb{CP}^1$  such that  $w_\infty : int(cl(\tilde{F}_\infty)) \rightarrow \mathbb{CP}^1 - [a_1, a_2]$  is a biholomorphism;  
 (5)  $\tilde{m}_\infty$  is the unique zero of  $w_\infty|_{int(cl(\tilde{F}_\infty))}$  and is simple;  $p_\infty(\tilde{m}_\infty) = \overline{m}^b$ .

*Proof.* The proof of Theorem 7.9.15 goes through without modification to give (1)–(3).

(4a) As in the proof of Theorem 7.9.15(4), we can define the degree for maps of pairs  $(cl(\tilde{F}_\infty), \partial cl(\tilde{F}_\infty)) \rightarrow (\mathbb{CP}^1, \mathbb{R}^+)$ . The degree of  $w_\infty$  is 1 because it has a unique pole of order 1. The order of the pole at the positive puncture can be computed from the winding number of  $f_{ab}$ , which is the smallest one for a positive eigenvalue by Lemma 7.8.4. Then  $w_\infty$  can have no branch points in the interior of  $cl(\tilde{F}_\infty)$ . (4b) is similar. (5) follows from (4a) and (4b).  $\square$

*Cases (c), (d') and (d'').* When the sequence  $\{b_i\}$  is unbounded, there are surfaces  $\tilde{F}_i^*$ ,  $\star \in \{m, b\}$ , with embeddings  $\iota_i^* : \tilde{F}_i^* \rightarrow \tilde{F}_i$ , such that  $\tilde{F}_i^*$  converges to  $\tilde{F}_\infty^*$ . Let  $p_i^m : \tilde{F}_i^m \rightarrow B_-$  be the restriction of  $p_i$  and let  $p_i^b : \tilde{F}_i^b \rightarrow B^*$ ,  $\star = \emptyset$  or  $'$ , be the composition of an  $s$ -translation and  $p_i|_{\tilde{F}_i^b}$  so that  $s$ -coordinate of  $p_i^b(b_i)$  is zero.

Let  $w_i$  be as in Definition 7.9.7. Let

$$K_i^b = (p_i \circ \iota_i^b)^{-1}(B_- \cap \{s(b_i) - 1 \leq s \leq s(b_i) + 1\}) \subset \tilde{F}_i^b$$

and  $C_i^b = \sup_{z \in K_i^b} |w_i(\iota_i^b(z))|$ . Then we set  $w_i^m = w_i \circ \iota_i^m$  and  $w_i^b = (w_i \circ \iota_i^b)/C_i^b$ .

**Lemma 7.9.17.** *For every compact set  $K \subset B^*$ , there is a constant  $C_K > 0$  such that  $|w_i^*(z)| < C_K$ , and  $|(w_i^*)'(z)| < C_K$  for all  $z \in (p_i^*)^{-1}(K)$  and all  $i \in \mathbb{N}$ . Here  $\star \in \{m, b\}$  and  $\star \in \{\emptyset, ', -\}$ , as appropriate.*

*Proof.* Similar to Lemma 7.9.14.  $\square$

Lemma 7.9.17 implies that the limits  $w_\infty^m$  and  $w_\infty^b$  exist. The following lemma gives the behavior of  $w_\infty^m$  and  $w_\infty^b$  near the punctures.

**Lemma 7.9.18.**

- (1) Let  $u : \mathbb{R}^+ \times (\mathbb{R}/\pi\mathbb{Z}) \rightarrow \mathbb{C}^\times$  be a finite energy holomorphic map. If the map  $t \mapsto u(s, t)$  has degree one for some (and therefore all)  $s \in \mathbb{R}^+$ , then  $\lim_{s \rightarrow +\infty} |u_\infty(s, t)| = +\infty$  and  $\lim_{s \rightarrow +\infty} \frac{u_\infty(s, t)}{|u_\infty(s, t)|} = ce^{\pi it}$  with  $c \neq 0$ .  
 (2) Let  $u : \mathbb{R}^- \times (\mathbb{R}/\pi\mathbb{Z}) \rightarrow \mathbb{C}^\times$  be a finite energy holomorphic map. If the map  $t \mapsto u(s, t)$  has degree one for some (and therefore all)  $s \in \mathbb{R}^-$ , then  $\lim_{s \rightarrow -\infty} |u_\infty(s, t)| = 0$ .

*Proof.* (1) Let us view  $u$  as a map  $\mathbb{R}^+ \times (\mathbb{R}/\pi\mathbb{Z}) \rightarrow \mathbb{R} \times (\mathbb{R}/\pi\mathbb{Z})$ . As in the proof of Lemma 7.9.14, since  $u$  has finite energy, it has bounded derivative. Let  $u_n(s, t) = u(s + k_n, t)$ , where  $k_n \in \mathbb{R}^+$  and  $\lim_{n \rightarrow +\infty} k_n = +\infty$ . The sequence

$u_n$  has uniformly bounded derivative and converges to a finite energy holomorphic map

$$u_\infty : \mathbb{R} \times (\mathbb{R}/\pi\mathbb{Z}) \rightarrow \mathbb{R} \times (\mathbb{R}/\pi\mathbb{Z}).$$

Such a holomorphic map is of the form  $u_\infty(s, t) = (s + a, t + b)$ , where  $a, b$  are constants. This implies (1). (2) is similar and is left to the reader.  $\square$

Case (c).

**Theorem 7.9.19.** *Suppose  $\lim_{i \rightarrow \infty} s(b_i) = +\infty$ . Then  $(w_i^m)_{i \in \mathbb{N}}$  converges to a holomorphic map  $w_\infty^m : \tilde{F}_\infty^m \rightarrow \mathbb{C}$  such that:*

- (1)  $w_\infty^m(\partial \tilde{F}_\infty^m) \subset \mathbb{R}^+$ ;
- (2) at the positive puncture  $\mathfrak{q}_+^m$ ,

$$\lim_{s \rightarrow +\infty} |w_\infty^m(s, t)| = +\infty, \quad \lim_{s \rightarrow +\infty} \frac{w_\infty^m(s, t)}{|w_\infty^m(s, t)|} = f(t),$$

where  $f$  is a normalized eigenfunction of the asymptotic operator at  $\delta_0$  with winding number one;

- (3) at the negative puncture  $\mathfrak{q}_-^m$ ,  $\lim_{s \rightarrow -\infty} w_\infty^m(s, t) = c \in \mathbb{R}^+$ ;
- (4)  $w_\infty^m$  extends to a holomorphic map  $w_\infty^m : cl(\tilde{F}_\infty^m) \rightarrow \mathbb{CP}^1$  such that  $w_\infty^m : int(cl(\tilde{F}_\infty^m)) \rightarrow \mathbb{CP}^1 - [a_1, a_2]$  is a biholomorphism; and
- (5)  $\overline{m}^b$  is the unique zero of  $w_\infty^m|_{int(cl(\tilde{F}_\infty^m))}$  and is simple.

Also  $(w_i^b)_{i \in \mathbb{N}}$  converges to a holomorphic map  $w_\infty^b : \tilde{F}_\infty^b \rightarrow \mathbb{C}$  such that:

- (6) at the positive puncture  $\mathfrak{q}_+^b$ ,

$$\lim_{s \rightarrow +\infty} |w_\infty^b(s, t)| = +\infty, \quad \lim_{s \rightarrow +\infty} \frac{w_\infty^b(s, t)}{|w_\infty^b(s, t)|} = f_{ab}(t);$$

- (7) at the negative puncture  $\mathfrak{q}_{-,1}^b$  that connects to  $\mathfrak{q}_+^m$ ,  $\lim_{s \rightarrow -\infty} w_\infty^b(s, t) = 0$ ;
- (8) at the other negative puncture  $\mathfrak{q}_{-,2}^b$ ,  $\lim_{s \rightarrow -\infty} w_\infty^b(s, t) = c \in \mathbb{R}^+$ ;
- (9) at the punctures  $\mathfrak{q}_+^m$  and  $\mathfrak{q}_{-,1}^b$ ,  $\lim_{s \rightarrow +\infty} \frac{w_\infty^m(s, t)}{|w_\infty^m(s, t)|} = \lim_{s \rightarrow -\infty} \frac{w_\infty^b(s, t)}{|w_\infty^b(s, t)|}$ ;
- (10)  $w_\infty^b$  extends to a biholomorphism  $w_\infty^b : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ ; and
- (11)  $\mathfrak{q}_{-,1}^b$  is the unique zero of  $w_\infty^b$  and is simple.

*Proof.* The proof of Theorem 7.9.15 goes through with little modification, in view of Lemma 7.9.18. We remark that (8) is a consequence of Convention 6.6.4 and the proof technique of Lemma 6.6.5.  $\square$

Cases (d') and (d'').

**Theorem 7.9.20.** *Suppose  $\lim_{i \rightarrow \infty} s(b_i) = -\infty$ . Then  $(w_i^m)_{i \in \mathbb{N}}$  converges to a holomorphic map  $w_\infty^m : \tilde{F}_\infty^m \rightarrow \mathbb{C}$  such that:*

- (1)  $w_\infty^m(\partial \tilde{F}_\infty^m) \subset \mathbb{R}^+$ ;

(2) at the positive puncture  $\mathfrak{q}_+^{\mathfrak{m}}$ ,

$$\lim_{s \rightarrow +\infty} |w_\infty^{\mathfrak{m}}(s, t)| = +\infty, \quad \lim_{s \rightarrow +\infty} \frac{w_\infty^{\mathfrak{m}}(s, t)}{|w_\infty^{\mathfrak{m}}(s, t)|} = f_{ab}(t);$$

(3) at the negative punctures  $\mathfrak{q}_{-,j}^{\mathfrak{m}}$ ,  $j = 1, 2$ ,  $\lim_{s \rightarrow -\infty} w_\infty^{\mathfrak{m}}(s, t) = c_j \in \mathbb{R}^+$ ;

(4)  $w_\infty^{\mathfrak{m}}$  extends to a holomorphic map  $w_\infty^{\mathfrak{m}} : cl(\tilde{F}_\infty^{\mathfrak{m}}) \rightarrow \mathbb{CP}^1$  such that  $w_\infty^{\mathfrak{m}} : int(cl(\tilde{F}_\infty^{\mathfrak{m}})) \rightarrow \mathbb{CP}^1 - [a_1, a_2]$  is a biholomorphism;

(5)  $\tilde{\mathfrak{m}}_\infty$  is the unique zero of  $w_\infty^{\mathfrak{m}}|_{int(cl(\tilde{F}_\infty^{\mathfrak{m}}))}$  and is simple.

Also  $(w_i^b)_{i \in \mathbb{N}}$  converges to a constant map  $w_\infty^b : \tilde{F}_\infty^b \rightarrow \mathbb{C}$ .

**7.10. Involution lemmas.** In this subsection we collect some lemmas on holomorphic maps between Riemann surfaces with anti-holomorphic involutions. These lemmas, collectively referred to as the *involution lemmas*, will play an important role in Section 7.11 and in the sequel [CGH2].

Our starting point is the following observation, whose proof is straightforward.

**Observation 7.10.1.** *Let  $\Sigma_1, \Sigma_2$  be Riemann surfaces with anti-holomorphic involutions  $\iota_1, \iota_2$ , respectively. If  $f : \Sigma_1 \rightarrow \Sigma_2$  is a holomorphic map, then  $\tilde{f} := \iota_2 \circ f \circ \iota_1$  is also holomorphic. Moreover, if  $f = \tilde{f}$ , then  $f(\text{Fix}(\iota_1)) \subset \text{Fix}(\iota_2)$ , where  $\text{Fix}(\iota_i)$  is the fixed point set of  $\iota_i$ .*

There are four versions of the involution lemma; the first two will be used in this paper and the last two only in the sequel [CGH2]. We start by introducing some common notation which will be used in all four versions: For  $i = 1, 2$ , the Riemann surface  $\Sigma_i$  is an open subset of  $\mathbb{CP}^1$  which is invariant under complex conjugation and has finitely generated fundamental group; moreover no component of  $\mathbb{CP}^1 - \Sigma_i$  is a single point. The complex conjugation on  $\mathbb{CP}^1$  restricts to an anti-holomorphic involution  $\iota_i : \Sigma_i \rightarrow \Sigma_i$ . On each  $\Sigma_i$  we fix “radial rays”

$$\mathcal{R}_i = \Sigma_i \cap (\mathbb{R}^{\leq 0} \cup \{\infty\}).$$

The *asymptotic marker*  $\dot{\mathcal{R}}_i(0)$  is the connected component of  $T_0\mathbb{RP}^1 - \{0\}$  (i.e., a tangent half-line) consisting of vectors with negative  $\partial_x$ -component; similarly, the asymptotic marker  $\dot{\mathcal{R}}_i(\infty)$  is the component of  $T_\infty\mathbb{RP}^1 - \{0\}$  that is mapped to  $\dot{\mathcal{R}}_i(0)$  under the inversion  $z \mapsto \frac{1}{z}$ . The radial rays  $\mathcal{R}_i$  and their related asymptotic markers are invariant under the involution  $\iota_i$ . In this section we will use the notation  $\mathbb{D}$  for the open unit disk in  $\mathbb{C}$ , considered as a Riemann surface.

**Lemma 7.10.2.** *Given  $\Sigma_i$  as above, there is a compact Riemann surface with boundary  $\bar{\Sigma}_i$  with a biholomorphism  $\Sigma_i \xrightarrow{\sim} int(\bar{\Sigma}_i)$ . Moreover there is an anti-holomorphic involution  $\iota_i : \bar{\Sigma}_i \rightarrow \bar{\Sigma}_i$  such that the diagram*

$$\begin{array}{ccc} \Sigma_i & \xrightarrow{\quad} & \bar{\Sigma}_i \\ \downarrow \iota_i & & \downarrow \iota_i \\ \Sigma_i & \xrightarrow{\quad} & \bar{\Sigma}_i \end{array}$$

*commutes.*

*Sketch of proof.* We outline the proof of the first statement. Use the uniformization theorem to identify the universal cover of  $\Sigma = \Sigma_1$  or  $\Sigma_2$  with the open upper half space  $\mathbb{H}$ . Let  $G \subset PSL(2, \mathbb{R})$  be the deck transformation group of  $\mathbb{H}$  such that  $\mathbb{H}/G = \Sigma$ . If  $G$  is finitely generated, then  $(\partial\mathbb{H} - L)/G$  is a collection of boundary circles of  $\Sigma$ , where  $L \subset \partial\mathbb{H}$  is the limit set of  $G$ . Hence  $\bar{\Sigma} = (\mathbb{H} - L)/G$ .  $\square$

Note that  $\bar{\Sigma}_i$  will not necessarily be the closure of  $\Sigma_i$  in  $\mathbb{CP}^1$ . However, when referring to points in the interior of  $\bar{\Sigma}_i$ , we denote them by the corresponding point in  $\Sigma_i$ . In the same way we view the radial rays  $\mathcal{R}_i$  as subsets of  $\bar{\Sigma}_i$ , and the asymptotic markers as tangent half-lines to  $\bar{\Sigma}_i$ . Lemma 7.10.2 implies that they are invariant by the involution on  $\bar{\Sigma}_i$ .

The image of an asymptotic marker by a holomorphic function is defined by the differential at regular points. At singular points the local behavior of a holomorphic map still allows us to define the image of a tangent ray.

For the first version of the involution lemma, let

$$\Sigma_1 = \mathbb{CP}^1 - ([a_1, a_2] \cup \dots \cup [a_{2p-1}, a_{2p}]),$$

where  $a_i \in \mathbb{R}^+$  and  $a_1 < \dots < a_{2p}$ , and let  $\Sigma_2 = \mathbb{D}$ . We write  $\partial\bar{\Sigma}_1 = \partial_1\bar{\Sigma}_1 \sqcup \dots \sqcup \partial_p\bar{\Sigma}_1$ , where  $\partial_i\bar{\Sigma}_1$ ,  $i = 1, \dots, p$ , corresponds to the slit  $[a_{2i-1}, a_{2i}]$ .

**Lemma 7.10.3** (Involution Lemma, Version 1). *Let  $f : \bar{\Sigma}_1 \rightarrow \bar{\Sigma}_2$  be a holomorphic map which is a  $q$ -fold branched cover with  $q \geq p$ , such that:*

- (i)  $f(\partial_i\bar{\Sigma}_1) = \partial\bar{\Sigma}_2$ ,  $i = 1, \dots, p$ ;
- (ii)  $f^{-1}(0) = \{\infty\}$ ; and
- (iii)  $f(0) \in \mathcal{R}_2$ .

*Then  $f$  maps  $\text{Fix}(\iota_1)$  to  $\text{Fix}(\iota_2)$  and  $\dot{\mathcal{R}}_1(\infty)$  to  $\dot{\mathcal{R}}_2(0)$ .*

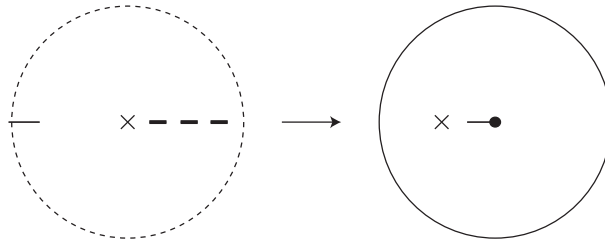


FIGURE 5. The map  $f|_{\Sigma_1}$  in Lemma 7.10.3. The left-hand side is  $\Sigma_1$ , where the point at infinity is represented by the dotted circle, and the right-hand side is  $\Sigma_2$ . The thicker lines on left-hand side are the slits  $[a_{2i-1}, a_{2i}]$  and the thinner lines are the asymptotic markers. The cross on the left is the origin and the cross on the right is  $f(0) \in \mathcal{R}_2$ . Lemma 7.10.3 states that  $f$  maps the asymptotic marker  $\dot{\mathcal{R}}_1(\infty)$  at  $\infty \in \Sigma_1$  to the asymptotic marker  $\dot{\mathcal{R}}_2(0)$  at  $0 \in \Sigma_2$ .

*Proof.* We claim that  $f = \tilde{f} = \iota_2 \circ f \circ \iota_1$ . To compare  $f$  and  $\tilde{f}$ , we consider their quotient  $Q = f/\tilde{f}$ . We observe three facts:

- (1)  $Q$  has no poles, since  $f$  and  $\tilde{f}$  have zeros only at  $\infty$  and their orders agree.
- (2)  $|Q(z)| = 1$  for all  $z \in \partial\overline{\Sigma}_1$ , so the maximum modulus theorem implies that  $Q(\overline{\Sigma}_1) \subset \overline{\mathbb{D}} = \{|z| \leq 1\}$ .
- (3) The degree of  $Q|_{\partial_i\overline{\Sigma}_1}$ , viewed as a map to  $S^1 = \partial\mathbb{D}$ , is zero, since

$$\deg(f|_{\partial_i\overline{\Sigma}_1}) = \deg(\tilde{f}|_{\partial_i\overline{\Sigma}_1}).$$

If  $Q$  is not constant, then by (3) there must be a branch point of  $Q$  along  $\partial_i\overline{\Sigma}_1$ . In particular,  $Q(\overline{\Sigma}_1) \not\subset \mathbb{D}$ , which contradicts (2). Hence  $Q$  is a constant map and  $f = c\tilde{f}$  for some  $c \in \mathbb{C} - \{0\}$ . Now (iii) implies that  $c = 1$ , and we have  $f = \tilde{f}$ .

Finally we apply Observation 7.10.1 to conclude that  $f$  maps  $\text{Fix}(\iota_1)$  to  $\text{Fix}(\iota_2)$  and, by (iii), maps  $\dot{\mathcal{R}}_1(\infty)$  to  $\dot{\mathcal{R}}_2(0)$ .  $\square$

The proofs of the other versions of the involution lemma are similar, and will be omitted.

**Lemma 7.10.4** (Involution Lemma, Version 2). *Let  $\Sigma_1 = \Sigma_2 = \mathbb{CP}^1$  and  $a_i \in \mathbb{R}^{\geq 0}$  with  $a_1 = 0 < a_2 < \dots < a_p$ . Assume  $f : \Sigma_1 \rightarrow \Sigma_2$  is a holomorphic map which is a  $q$ -fold branched cover with  $q \geq p$ , such that:*

- (i)  $f^{-1}(\infty) = \{a_1, \dots, a_p\}$ ;
- (ii)  $f^{-1}(0) = \{\infty\}$ ; and
- (iii)  $f$  maps  $\dot{\mathcal{R}}_1(0)$  to  $\dot{\mathcal{R}}_2(\infty)$ .

*Then  $f$  maps  $\text{Fix}(\iota_1)$  to  $\text{Fix}(\iota_2)$  and  $\dot{\mathcal{R}}_1(\infty)$  to  $\dot{\mathcal{R}}_2(0)$ .*

**Lemma 7.10.5** (Involution Lemma, Version 3). *Let  $\Sigma_1 = \Sigma_2 = \mathbb{D}$  and  $a_i \in \mathbb{R}^{\geq 0}$  with  $a_1 = 0 < a_2 < \dots < a_p < 1$ . Assume  $f : \overline{\Sigma}_1 \rightarrow \overline{\Sigma}_2$  is a holomorphic map which is a  $q$ -fold branched cover with  $q \geq p$ , such that:*

- (i)  $f^{-1}(0) = \{a_1, \dots, a_p\}$ ;
- (ii)  $f^{-1}(\partial\overline{\Sigma}_1) = \partial\overline{\Sigma}_2$ ; and
- (iii)  $f$  maps  $\dot{\mathcal{R}}_1(0)$  to  $\dot{\mathcal{R}}_2(0)$ .

*Then  $f$  maps  $\text{Fix}(\iota_1)$  to  $\text{Fix}(\iota_2)$ . Moreover  $f(-1) = -1$ .*

For the fourth version of the involution lemma, we consider  $\Sigma_1 = \mathbb{D} - ([a_1, a_2] \cup \dots \cup [a_{2p-1}, a_{2p}])$ , where  $a_i \in (0, 1)$  with  $a_1 < \dots < a_{2p}$ , and  $\Sigma_2 = \{R < |z| < 1\}$ , where  $0 < R < 1$ . We write  $\partial\overline{\Sigma}_1 = \partial_0\overline{\Sigma}_1 \sqcup \dots \sqcup \partial_p\overline{\Sigma}_1$ , where  $\partial_0\overline{\Sigma}_1$  is the boundary component which can be identified with  $\{|z| = 1\}$  and  $\partial_i\overline{\Sigma}_1$ ,  $i = 1, \dots, p$ , corresponds to the slit  $[a_{2i-1}, a_{2i}]$ .

**Lemma 7.10.6** (Involution Lemma, Version 4). *Let  $f : \overline{\Sigma}_1 \rightarrow \overline{\Sigma}_2$  be a holomorphic map which is a  $q$ -fold branched cover with  $q \geq p$ , such that:*

- (i)  $f(\partial_i\overline{\Sigma}_1) = \{|z| = 1\}$  when  $i \in \mathcal{I} \subset \{1, \dots, p\}$ ;
- (ii)  $f(\partial_i\overline{\Sigma}_1) = \{|z| = R\}$  when  $i = 0$  or  $i \notin \mathcal{I}$ ; and
- (iii)  $f(0) \in \mathcal{R}_2$ .

*Then  $f$  maps  $\text{Fix}(\iota_1)$  to  $\text{Fix}(\iota_2)$ . Moreover  $f(-1) = -R$ .*



See Figure 6.

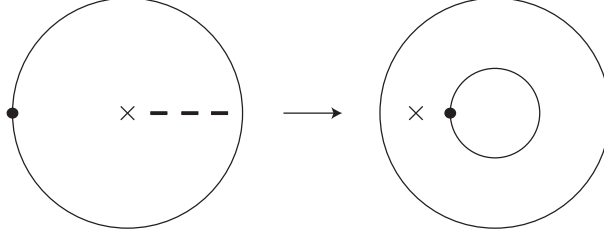


FIGURE 6. The map  $f|_{\Sigma_1}$  in Lemma 7.10.6. The left-hand side is  $\Sigma_1$  and the right-hand side is  $\Sigma_2$ . The thick lines on the left-hand side are the slits  $[a_{2i-1}, a_{2i}]$ . The cross on the left is the origin and the cross on the right is  $f(0) \in \mathcal{R}_2$ . The dots represent  $-1 \in \partial\Sigma_1$  and  $f(-1) \in \partial\Sigma_2$ . Lemma 7.10.6 states that  $f(-1) = -R$ .

**7.11. Elimination of some cases.** We are now in a position to eliminate some of possibilities that appear in Theorem 7.7.1:

**Theorem 7.11.1.** *Suppose  $m \gg 0$ ,  $\varepsilon, \delta > 0$  are sufficiently small,  $\bar{u}_\infty \in \partial\mathcal{M}_{\bar{\mathfrak{m}}}^i(\varepsilon, \delta, p)$ , and  $\bar{v}'_0 \neq \emptyset$ .*

- (i) *If  $i = 2$ , then the 3-level subbuilding described in Theorem 7.7.3 does not occur.*
- (ii) *If  $i = 3$ , then Cases (2)–(6) of Theorem 7.7.1 do not occur.*

We briefly sketch the idea of the proof. In all the cases that are eliminated by Theorem 7.11.1, the (unique) component of  $\cup_{j=1}^a \bar{v}_j^\# \subset \bar{u}_\infty$  which is asymptotic to a multiple of  $\delta_0$  at the negative end has ECH index  $I = 1$ .

Suppose that, for  $m \gg 0$ , there is a sequence of holomorphic curves which converges to a configuration  $\bar{u}_\infty$  that we want to exclude. In Section 7.9, we applied a rescaling argument to construct a holomorphic building which keeps track of how the limit  $\bar{u}_\infty$  is approached; this is similar to the layer structures of Ionel-Parker [IP1, Section 7]. In the simplest case, this building is a holomorphic map

$w_\infty : B_- \rightarrow \mathbb{C}$  which satisfies the asymptotic condition  $\lim_{s \rightarrow +\infty} \frac{w_\infty(s, t)}{|w_\infty(s, t)|} = f_{ab}(t)$ , where  $f_{ab}(t)$  is a normalized asymptotic eigenfunction of an  $I = 1$  curve with a negative end asymptotic to  $\delta_0$ . The condition  $I = 1$  is used as follows: since there are only finitely many  $I = 1$  curves with negative ends asymptotic to  $\delta_0$ , we may assume that  $-1 \notin \{f_{ab}(\frac{3}{2})\}$  as explained in Remark 7.8.8.

**Remark 7.11.2.** In this subsection we identify  $cl(B_-) \simeq \bar{\mathbb{D}}$  so that  $\mathfrak{p}_+$  corresponds to 0 and  $\mathfrak{p}_-$  corresponds to 1. There is an anti-holomorphic involution  $\iota$  on  $B_-$  that fixes the half-line  $\{t = \frac{3}{2}\}$ , and  $\{t = \frac{3}{2}\}$  corresponds to the radial ray  $\mathcal{R} = \bar{\mathbb{D}} \cap \mathbb{R}^{\leq 0}$  by Observation 7.10.1. In particular,  $\bar{\mathfrak{m}}^b$  is mapped to a point on  $\mathcal{R}$ .

Similarly we identify  $cl(B') \simeq \mathbb{CP}^1$  so that  $\mathfrak{p}_+$  corresponds to 0,  $\mathfrak{p}_-$  corresponds to  $\infty$ , and  $\{t = \frac{3}{2}\}$  corresponds to the radial ray  $\mathbb{R}^{\leq 0}$ ,

We now use the involutions lemmas from Section 7.10 to obtain a contradiction. By the involution lemmas and the symmetric placement of the basepoint  $\overline{m}^b$ , we obtain that  $w_\infty \circ \iota = i \circ w_\infty$ . Hence  $\lim_{s \rightarrow +\infty} \frac{w_\infty(s, \frac{3}{2})}{|w_\infty(s, \frac{3}{2})|} = -1$ , which contradicts  $-1 \notin \{f_{ab}(\frac{3}{2})\}$ .

**7.11.1. Proof of Theorem 7.11.1.** In this subsection we use limiting arguments in which  $m \rightarrow \infty$  and  $\overline{h}_m \rightarrow \overline{h}_\infty$ ; see Section 7.8.4. Hence many of the almost complex structures and moduli spaces will have an additional subscript  $m$ , where  $m = \infty$  is also a possibility. For example,  $\overline{\mathcal{J}}'_m$  and  $\overline{\mathcal{J}}_{-,m}$  refer to  $\overline{\mathcal{J}}'$  and  $\overline{\mathcal{J}}_-$  with respect to  $m$ .

Let  $\overline{\mathcal{J}}'_\infty \in (\overline{\mathcal{J}}'_\infty)_\star^{reg}$  and let  $\overline{\mathcal{J}}'_m \in (\overline{\mathcal{J}}'_m)_\star^{reg}$  be a nearby almost complex structure with respect to the integer  $m \gg 0$ . Let  $\overline{\mathcal{J}}_{-,m} \in \overline{\mathcal{J}}_{-,m}^{reg}$  be an almost complex structure which restricts to  $\overline{\mathcal{J}}'_m$  and let  $\overline{\mathcal{J}}_{-,m}^\diamond$  be  $(\varepsilon, U)$ -close to  $\overline{\mathcal{J}}_{-,m}$ .

We will treat Theorem 7.7.1 in detail and leave Theorem 7.7.3 to the reader. Suppose that, for a sequence  $m_i \rightarrow \infty$ , there sequences  $\overline{u}_{ij}$  of  $\overline{\mathcal{J}}_{-,m_i}^\diamond$ -holomorphic curves which converge to a  $\overline{\mathcal{J}}_{-,m_i}^\diamond$ -holomorphic building  $\overline{u}_{i\infty}$  falling into one of Cases (2)–(6).

*Elimination of Case (2).* Suppose for each  $i$  the sequence  $\overline{u}_{ij}$  converges a building  $\overline{u}_{i\infty}$  satisfying Case (2). By Theorem 7.9.15, we obtain a holomorphic map  $w_\infty : cl(B_-) \rightarrow \mathbb{CP}^1$ , whose restriction to  $int(cl(B_-))$  is a biholomorphism onto its image.

We apply the Involution Lemma 7.10.3 to obtain a contradiction: Let  $\overline{\Sigma}_1$  be the compactification of  $\Sigma_1 = \mathbb{CP}^1 - [a_1, a_2]$  and let  $\overline{\Sigma}_2 = cl(B_-)$  be identified with  $\mathbb{D}$  as in Remark 7.11.2. Let  $f : \overline{\Sigma}_1 \rightarrow \overline{\Sigma}_2$  be the extension of  $(w_\infty|_{int(cl(B_-))})^{-1}$ . Such an extension exists because  $\Sigma_1$  is biholomorphic to the open unit disk and biholomorphisms of the open unit disk extend continuously to the boundary.

By Lemma 7.10.3,  $f$  maps  $\dot{\mathcal{R}}_1(\infty)$  to  $\dot{\mathcal{R}}_2(0)$  and, conversely,  $w_\infty$  maps  $\dot{\mathcal{R}}_2(0)$  to  $\dot{\mathcal{R}}_1(\infty)$ . Since the asymptotic marker  $\dot{\mathcal{R}}_2(0)$  in  $\overline{\Sigma}_2$  corresponds to the asymptotic marker  $\{t = \frac{3}{2}\}$  for  $p_+ \in cl(B_-)$  by Remark 7.11.2,  $\dot{\mathcal{R}}_1(\infty)$  is a bad radial ray (in the sense of Definition 7.8.7) by Theorem 7.9.15(2). This contradicts Remark 7.8.8, so we have eliminated Case (2).

*Elimination of Cases (3) and (4).* We will treat Case (4); Case (3) is almost identical. Suppose for each  $i$  the sequence  $\overline{u}_{ij}$  converges to a building  $\overline{u}_{i\infty}$  satisfying Case (4). By Remark 7.7.2, for each  $i$ , the total number of branched points of  $\cup_{j=-b}^a \overline{v}_{j,i}'$  is one. If we exercise some care in choosing the diagonal sequence in Lemma 7.9.4, we can divide the argument for Case (4) further into Subcases (a), (b), (c), (d') and (d'') as in Section 7.9.8, depending on the behavior of the branch points of the maps  $\pi_{B_-} \circ \overline{v}_{0,i}'$ .

*Subcases (a) and (b).* By Theorem 7.9.16, we obtain holomorphic maps

$$w_\infty : cl(\tilde{F}_\infty) \rightarrow \mathbb{CP}^1, \quad p_\infty : \tilde{F}_\infty \rightarrow B_-,$$

where  $w_\infty|_{\text{int}(cl(\tilde{F}_\infty))}$  is a biholomorphism onto its image  $\Sigma_1 = \mathbb{CP}^1 - ([a_1, a_2] \cup [a_3, a_4])$  and  $p_\infty$  is a branched double cover with one branch point. Let  $\bar{\Sigma}_1$  be the compactification of  $\Sigma_1$  as in Lemma 7.10.2, and let  $\bar{\Sigma}_2 = cl(B_-)$  be identified with  $\bar{\mathbb{D}}$  as in Remark 7.11.2. We define  $f : \bar{\Sigma}_1 \rightarrow \bar{\Sigma}_2$  as the extension of  $p_\infty \circ (w_\infty|_{\text{int}(cl(\tilde{F}_\infty))})^{-1}$ . Such an extension exists because  $\Sigma_1$  is biholomorphic to an open annulus, and biholomorphisms of the unit annulus always extend to the boundary. At this point we apply Lemma 7.10.3 as in Case (2) to obtain a contradiction. Case (b) is completely analogous and can be excluded in the same way.

*Subcase (c).* By Theorem 7.9.19, we obtain pairs of holomorphic maps

$$w_\infty^\star : cl(\tilde{F}_\infty^\star) \rightarrow \mathbb{CP}^1, \quad p_\infty^\star : \tilde{F}_\infty^\star \rightarrow B^\star,$$

where  $\star \in \{m, b\}$  and  $\star \in \{\emptyset, ', -\}$ , as appropriate. The map  $w_\infty^m$  restricts to a biholomorphism of  $\text{int}(cl(\tilde{F}_\infty^m))$  with  $\Sigma_1^m = \mathbb{CP}^1 - [a_1, a_2]$  and the map  $p_\infty^m$  is a degree 1 branched cover (i.e., a biholomorphism). Let  $\bar{\Sigma}_1^m$  be the compactification of  $\Sigma_1^m$ . Identify  $\bar{\Sigma}_2^m = cl(B_-)$  with  $\bar{\mathbb{D}}$  as in Remark 7.11.2. Let  $f^m : \bar{\Sigma}_1^m \rightarrow \bar{\Sigma}_2^m$  be the extension of  $p_\infty^m \circ (w_\infty^m|_{\text{int}(cl(\tilde{F}_\infty^m))})^{-1}$ . As in Case (2), Lemma 7.10.3 implies that  $f^m$  maps  $\dot{\mathcal{R}}_1^m(\infty)$  to  $\dot{\mathcal{R}}_2^m(0)$  and, conversely,  $w_\infty^m$  maps  $\dot{\mathcal{R}}_2^m(0)$  to  $\dot{\mathcal{R}}_1^m(\infty)$ .

Next we consider the “upper level”  $(w_\infty^b, p_\infty^b)$ . The map  $w_\infty^b : cl(\tilde{F}_\infty^b) \rightarrow \mathbb{CP}^1$  is a biholomorphism and the map  $p_\infty^b : \tilde{F}_\infty^b \rightarrow B'$  is a branched double cover with one branch point. We define  $f^b = p_\infty^b \circ (w_\infty^b)^{-1} : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ , using the identification  $cl(B') \simeq \mathbb{CP}^1$  from Remark 7.11.2. By Theorem 7.9.19(9) and the previous paragraph,  $f^b$  maps  $\dot{\mathcal{R}}_1(0)$  to  $\dot{\mathcal{R}}_2(\infty)$ . Then, by Lemma 7.10.4 and Theorem 7.9.19(7),(8),  $f^b$  maps  $\dot{\mathcal{R}}_1(\infty)$  to  $\dot{\mathcal{R}}_2(0)$ . As in Case (2), this is a contradiction because  $\dot{\mathcal{R}}_1(\infty)$  is a good radial ray.

*Subcases (d') and (d'').* By Theorem 7.9.20, we obtain a holomorphic map  $w_\infty^m : cl(\tilde{F}_\infty^m) \rightarrow \mathbb{CP}^1$  which restricts to a biholomorphism of  $\text{int}(cl(\tilde{F}_\infty^m))$  with  $\mathbb{CP}^1 - [a_1, a_2]$ . Then the proof proceeds as in Case (2).

*Elimination of Cases (5) and (6).* The limit configurations of Cases (5) and (6) must contain a connector over  $\delta_0$ . This implies that Theorem 7.9.19 applies, and we obtain pairs of holomorphic maps

$$w_\infty^\star : cl(\tilde{F}_\infty^\star) \rightarrow \mathbb{CP}^1, \quad p_\infty^\star : \tilde{F}_\infty^\star \rightarrow B^\star,$$

where  $\star \in \{m, b\}$ ,  $\star \in \{\emptyset, ', -\}$ ,  $cl(\tilde{F}_\infty^m) \simeq \bar{\mathbb{D}}$ , and  $cl(\tilde{F}_\infty^b) \simeq \mathbb{CP}^1$ . Then the argument of Case (4c) applies.

This completes the proof of Theorem 7.11.1.

**7.12. Proof of Lemma 7.2.3.** We begin with the following corollary of Theorem 7.11.1.

**Corollary 7.12.1.** *Suppose  $m \gg 0$  and  $\varepsilon, \delta > 0$  are sufficiently small constants. If  $\bar{u} \in \mathcal{M}_{\bar{m}}^2(\varepsilon, \delta, p)$  or  $\mathcal{M}_{\bar{m}}^{3,(r_0)}(\varepsilon, \delta, p)$ , then  $\text{Im}(\bar{u}) \cap K_{p,2\delta} \neq \emptyset$ . In the latter case,  $r_0 \gg 0$  and  $\varepsilon, \delta > 0$  are sufficiently small constants which depend on  $r_0$ .*

*Proof.* Arguing by contradiction, suppose there are sequences  $\varepsilon_i, \delta_i \rightarrow 0$  and  $\bar{u}_i \in \mathcal{M}_{\bar{m}}^{3,(r_0)}(\varepsilon_i, \delta_i, p)$  such that  $\text{Im}(\bar{u}_i) \cap K_{p,2\delta_i} \neq \emptyset$ . Then the limit  $\bar{u}_\infty \in \partial\mathcal{M}_{\bar{m}}^3(0, 0, p)$  of  $\bar{u}_i$  has a nontrivial  $\bar{v}_0$  component. By Theorems 7.7.1 and 7.11.1(ii),  $\bar{u}_\infty$  satisfies Case (1) of Theorem 7.7.1. Hence, for  $i \gg 0$ ,  $\bar{u}_i \in G(\mathfrak{P}_{(r_0)})$ , which is a contradiction. The case of  $\bar{u} \in \mathcal{M}_{\bar{m}}^2(\varepsilon, \delta, p)$  is easier and is a consequence of Theorems 7.7.3 and 7.11.1(i).  $\square$

*Proof of Lemma 7.2.3.* Suppose that  $\bar{u}_\infty \in \partial_1\mathcal{M}_{\bar{m}}^3$ . We are in the situation of Lemma 7.6.2. If  $\bar{v}_0$  is a degenerate  $\bar{W}_-$ -curve, then  $I(\bar{v}_0) \geq 4$  by Equation (7.6.6) since  $g \geq 1$ . Therefore degenerate  $\bar{W}_-$ -curves are ruled out by Constraint (i) of Theorem 7.7.1. Hence  $\bar{v}_0$  is a  $\bar{W}_-$ -curve and the other levels  $\bar{v}_j$ ,  $j \neq 0$ , are multi-sections of  $W'$  or  $W$ . In this case, the curve  $\bar{v}_0$  is simply-covered. Corollary 7.12.1 implies that the component through  $\bar{m}$  also intersects  $K_{p,2\delta}$ . Hence passing through  $\bar{m}$  is a generic condition and we must have  $\text{ind}(\bar{v}_0) \geq 2$  and  $I(\bar{v}_0) \geq 2$ . This gives us two options: either

- ( $\alpha$ )  $b = 0$ ,  $I(\bar{v}_a) = 1$ , and  $I(\bar{v}_j) = 0$ ,  $j = 1, \dots, a-1$ ; or
- ( $\beta$ )  $a = 0$ ,  $b = 1$ , and  $I(\bar{v}_{-1}) = 1$ .

Ghost components are not possible since each ghost component takes up  $\text{ind} \geq 2$  by Lemma 6.1.7. Hence  $\bar{u}_\infty \in A_1 \cup A_2$ .  $\square$

**7.13. Gluing.** The goal of this subsection is to prove Theorem 7.2.2.

**7.13.1. The moduli space  $\mathcal{N}$ .** Let  $\varepsilon = \frac{\pi}{m}$ , where  $m$  is a sufficiently large integer and let  $\eta_\varepsilon : [-\pi, \pi] \rightarrow \mathbb{R}$  be a smooth map such that:

- $\eta_\varepsilon(\theta) = \varepsilon$  for  $-\pi \leq \theta \leq \theta_1$ ;
- $\eta_\varepsilon(\theta) = 0$  for  $\theta_2 \leq \theta \leq \pi$ ; and
- $\eta_\varepsilon$  is monotonically decreasing for  $\theta_1 \leq \theta \leq \theta_2$ ;

for some  $-\pi < \theta_1 < \theta_2 < \pi$ .

We consider the usual coordinates  $(s, t) \in \mathbb{R} \times [0, 2]/(0 \sim 2)$  on  $B_-$  and parametrize  $\partial B_-$  in an orientation-preserving manner by a coordinate  $\theta \in (-\pi, \pi)$ . We also fix an identification of  $cl(B_-)$  with  $\mathbb{D}$  such that  $\mathfrak{p}_+$  is mapped to 0 and  $\mathfrak{p}_-$  is mapped to  $-1$ .

**Definition 7.13.1.** Let  $\mathcal{N} = \mathcal{N}_{\eta_\varepsilon}$  be the space of holomorphic maps  $w : B_- \rightarrow \mathbb{C}$  such that the following properties hold:

- (N<sub>1</sub>)  $w(e^{i\theta}) \in \mathbb{R}^+ \cdot e^{i\eta_\varepsilon(\theta)}$  for all  $\theta \in (-\pi, \pi)$ ;
- (N<sub>2</sub>)  $\lim_{s \rightarrow -\infty} |w(s, t) - c_1 e^{-\varepsilon(s+it-i)}| < \infty$  for some  $c_1 \in \mathbb{R}^+$ ; and
- (N<sub>3</sub>)  $\lim_{s \rightarrow +\infty} |w(s, t) - c_2 e^{\pi(s+it)}| < \infty$  for some  $c_2 \in \mathbb{C}^\times$ .

In particular:

- (N<sub>4</sub>)  $\deg(w) = 1$  away from the sector  $\{0 \leq \phi \leq \varepsilon\} \subset \mathbb{CP}^1$ ; and

(N<sub>5</sub>) after composing with the chosen identification  $cl(B_-) \cong \overline{\mathbb{D}}$ ,  $w$  extends continuously to a  $\overline{\mathbb{D}}$  so that  $w(0) = w(-1) = \infty$ .

Multiplication by a real constant gives an  $\mathbb{R}^+$ -action on  $\mathcal{N}$ .

Even though  $\mathcal{N}$  is the space we are interested in, it will be convenient for technical reasons to regard  $\mathcal{N}$  as an open subset of a vector space  $\tilde{\mathcal{N}}$  obtained by relaxing properties (N<sub>1</sub>)–(N<sub>3</sub>).

**Definition 7.13.2.** Let  $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}_{\eta_\varepsilon}$  be the space of holomorphic maps  $w : B_- \rightarrow \mathbb{C}$  such that the following properties hold:

- ( $\tilde{N}_1$ )  $w(e^{i\theta}) \in \mathbb{R} \cdot e^{i\eta_\varepsilon(\theta)}$  for all  $\theta \in (-\pi, \pi)$ ;
- ( $\tilde{N}_2$ )  $\lim_{s \rightarrow -\infty} |w(s, t) - c_1 e^{-\varepsilon(s+it-i)}| < \infty$  for some  $c_1 \in \mathbb{R}$ ; and
- ( $\tilde{N}_3$ )  $\lim_{s \rightarrow +\infty} |w(s, t) - c_2 e^{\pi(s+it)}| < \infty$  for some  $c_2 \in \mathbb{C}$ .

In order to compute the dimension of  $\tilde{\mathcal{N}}$ , we identify  $B_-$  with  $\overline{\mathbb{D}} - \{-1, 0\}$  and  $\tilde{\mathcal{N}}$  with the space of the holomorphic sections of a holomorphic line bundle  $E \rightarrow \overline{\mathbb{D}}$  with values in a real rank one subbundle  $F$  along  $\partial\overline{\mathbb{D}} - \{-1\}$ .

We construct the bundles  $E$  and  $F$  as follows. Consider a cover of  $\overline{\mathbb{D}}$  by three open sets

$$U_0 = \overline{\mathbb{D}} - \{0, 1\}, \quad U_1 = \{z \in \mathbb{D} \mid |z| < 1/3\}, \quad U_2 = \{z \in \overline{\mathbb{D}} \mid |z+1| < 1/3\}.$$

Over each open set we take a trivial line bundle  $E_i = \mathbb{C} \times U_i \rightarrow U_i$  and define the bundle  $E$  by gluing the bundles  $E_i$  via the transition maps

$$\begin{aligned} \psi_1 : E_0|_{U_0 \cap U_1} &\rightarrow E_1|_{U_0 \cap U_1}, & \psi_1(z, v) &= (z, zv), \\ \psi_2 : E_0|_{U_0 \cap U_2} &\rightarrow E_2|_{U_0 \cap U_2}, & \psi_2(z, v) &= \left(z, i \left(\frac{z+1}{-z+1}\right) v\right). \end{aligned}$$

Let  $\pi_{E_i} : E_i = \mathbb{C} \times U_i \rightarrow \mathbb{C}$  be the projections corresponding to the trivializations. If we parametrize  $\partial\overline{\mathbb{D}} - \{-1\}$  by  $\theta \in (-\pi, \pi)$ , the subbundle  $F$  is given, as a subbundle of  $E_0$ , by  $F(\theta) = \mathbb{R} \cdot e^{i\eta_\varepsilon(\theta)}$ .

It is convenient to view  $\Sigma = \overline{\mathbb{D}} - \{-1, 0\}$  as a surface with a negative strip-like end and a positive cylindrical end and  $E$  as a line bundle over  $\Sigma$ . Let  $(-\infty, -R) \times [0, 1]$  be a strip with coordinates  $(s, t)$ . We identify the strip-like end  $Z$  of  $\Sigma$  with  $(-\infty, -R) \times [0, 1]$  via the map

$$\phi : (-\infty, -R) \times [0, 1] \rightarrow \Sigma, \quad \phi(s, t) = \frac{e^{\pi(s+it)} - i}{e^{\pi(s+it)} + i}.$$

Here the fractional linear transformation  $B(\zeta) = \frac{\zeta-i}{\zeta+i}$  maps the upper half plane  $\mathbb{H}$  to the unit disk  $\mathbb{D}$  and 0 to  $-1$ . Observe that  $A(\zeta) = i(\frac{\zeta+1}{-\zeta+1})$  which appears in the definition of  $\psi_2$  is the inverse of  $B(\zeta)$ . Hence the gluing map  $\psi_2$  becomes

$$\psi_2((s, t), v) = ((s, t), e^{\pi(s+it)} v),$$

with respect to coordinates  $(s, t)$ .

The linear Cauchy–Riemann operator

$$D : W^{1,p}(E, F) \rightarrow L^p(T^{0,1}\Sigma \otimes_{\mathbb{C}} E)$$

is Fredholm for  $p > 2$  and its kernel consists of smooth holomorphic functions. We denote by  $H^0(E, F)$  its kernel and by  $H^1(E, F)$  its cokernel.

**Lemma 7.13.3.** *There is an identification  $H^0(E, F) \cong \tilde{\mathcal{N}}$  for every choice of  $\eta_\varepsilon$ .*

*Proof.* The isomorphism  $H^0(E, F) \cong \tilde{\mathcal{N}}$  associates to a holomorphic section  $\xi \in H^0(E, F)$  the holomorphic function  $\pi_{E_0} \circ \xi : \Sigma \rightarrow \mathbb{C}$ , i.e.,  $\pi_{E_0} \circ \xi$  is obtained by writing  $\xi|_{U_0}$  with respect to the trivialization of  $E_0$ . On the negative end  $Z$  we can write

$$\pi_{E_2} \circ \xi(s, t) = \sum_{n \geq 1} c_n e^{(n\pi - \varepsilon)(s+it) + i\varepsilon},$$

since  $\pi_{E_2} \circ \xi$  is holomorphic. By applying the transition function  $\psi_2^{-1}$ , we obtain that the leading term of  $\pi_{E_0} \circ \xi$  on  $Z$  is  $e^{-\varepsilon(s+it) + i\varepsilon}$ , which is condition  $(\tilde{\mathcal{N}}_2)$  in Definition 7.13.2. For a similar reason  $\pi_{E_0} \circ \xi$  has a pole at 0 of order at most 1.  $\square$

We will consider also the compactified surface  $\tilde{\Sigma}$  obtained by adding the “segment at infinity” to the strip-like end  $Z$ . Alternatively,  $\tilde{\Sigma}$  admits an identification with the truncated surface  $\Sigma - Z$ . Let  $\check{E} \rightarrow \tilde{\Sigma}$  be the line bundle obtained by extending  $E \rightarrow \Sigma$ .

**Lemma 7.13.4.** *ind  $D = 3$  for every choice of  $\eta_\varepsilon$ .*

*Proof.* We decompose  $\partial\tilde{\Sigma} = cl(\partial\Sigma) \cup (\partial\tilde{\Sigma} - \partial\Sigma)$ , where  $\partial\tilde{\Sigma} - \partial\Sigma$  corresponds to the “segment at infinity” of the negative strip-like end  $Z$  and  $cl(\partial\Sigma)$  is the closure of  $\partial\Sigma$  in  $\partial\tilde{\Sigma}$ . We parametrize  $cl(\partial\Sigma)$  by  $\theta \in [-\pi, \pi]$  and  $\partial\tilde{\Sigma} - \partial\Sigma$  by  $\theta' \in [0, 1]$  in a manner compatible with the orientation of  $\tilde{\Sigma}$  induced by the complex structure on  $\Sigma$ . In particular,  $\theta = -\pi$  is identified with  $\theta' = 1$  and  $\theta = \pi$  is identified with  $\theta' = 0$ .

We define a trivialization  $\tau$  of  $\check{E}|_{\partial\tilde{\Sigma}}$  by:

$$\begin{cases} \tau(\theta) = e^{i\eta_\varepsilon(\theta)} & \text{along } cl(\partial\Sigma), \text{ with respect to the trivialization of } E_0, \\ \tau(\theta') = -e^{i(\pi+\varepsilon)\theta'} & \text{along } \partial\tilde{\Sigma} - \partial\Sigma, \text{ with respect to the trivialization of } E_2. \end{cases}$$

We also define a Lagrangian subbundle  $\check{F} \subset \check{E}|_{\partial\tilde{\Sigma}}$  by  $\check{F}|_{\partial\Sigma} = F$  and rotating it in the counterclockwise direction by the minimal amount on  $\partial\tilde{\Sigma} - \partial\Sigma$ . This means:

$$\begin{cases} F(\theta) = \mathbb{R} \cdot e^{i\eta_\varepsilon(\theta)} & \text{along } cl(\partial\Sigma), \text{ with respect to the trivialization of } E_0, \\ F(\theta') = \mathbb{R} \cdot e^{i\varepsilon\theta'} & \text{along } \partial\tilde{\Sigma} - \partial\Sigma, \text{ with respect to the trivialization of } E_2. \end{cases}$$

By the doubling argument of Theorem 5.5.1, Lemma 5.5.3 (or, rather, its proof) and the formula for the index of the Cauchy–Riemann operator on line bundles over punctured surfaces (see for example [We3, Formula 2.1]), the index of  $D$  is

$$\text{ind } D = \chi(\mathbb{D}) + \mu_\tau(\check{F}) + 2c_1(\check{E}, \tau) - 1.$$

From the explicit definitions of  $\tau$  and  $\check{F}$  one computes that  $\mu_\tau(\check{F}) = -1$  and  $c_1(\check{E}, \tau) = 2$ . Hence  $\text{ind } D = 3$ .  $\square$

**Lemma 7.13.5.** *The operator  $D$  is surjective for every choice of  $\eta_\varepsilon$ .*

*Proof.* In view of Theorem 5.5.1, the surjectivity of  $D$  follows from [We3, Formula 2.5] and [We3, Proposition 2.2(2)] applied to the double of  $D$ .  $\square$

**Corollary 7.13.6.** *The real dimension of  $\tilde{\mathcal{N}}$  is 3 for every choice of  $\eta_\varepsilon$ .*

7.13.2. *The maps  $\mathfrak{E}$  and  $\mathfrak{F}$ .* We define an  $\mathbb{R}$ -linear map:

$$\begin{aligned}\mathfrak{E} : \tilde{\mathcal{N}}_{\eta_\varepsilon} &\rightarrow \mathbb{R} \times \mathbb{C}, \\ w &\mapsto (c_1, c_2),\end{aligned}$$

where  $c_1, c_2$  are the coefficients from  $(\tilde{\mathcal{N}}_1)$  and  $(\tilde{\mathcal{N}}_2)$  of Definition 7.13.2.

**Lemma 7.13.7.** *The map  $\mathfrak{E}$  is an isomorphism.*

Hence  $\mathfrak{E}(\mathcal{N})$  is an open positive cone contained in  $\mathbb{R}^+ \times \mathbb{C}^\times$ .

*Proof.* It suffices to check that  $\mathfrak{E}$  is injective. This follows from the winding number argument of [Se2, Lemma 11.5], which we sketch: First observe that  $\ker \mathfrak{E}$  consists of holomorphic functions  $u : \overline{\mathbb{D}} \rightarrow \mathbb{C}$  such that  $u(-1) = 0$  and  $u(e^{i\theta}) \in \mathbb{R} \cdot e^{\eta_\varepsilon(\theta)}$  for all  $\theta \in (-\pi, \pi)$ . Suppose that  $u \in \ker \mathfrak{E}$  and  $u \neq 0$ . Given  $z \in \overline{\mathbb{D}}$ , let  $\nu(z)$  be the order of zero at  $z$ , with the convention that  $\nu(z) = 0$  if  $u(z) \neq 0$ . Then, by analyzing the boundary and asymptotic conditions of  $u$ , we obtain

$$(7.13.1) \quad \sum_{z \in \text{int}(\Sigma)} \nu(z) + \frac{1}{2} \sum_{z \in \partial \Sigma} \nu(z) < 0,$$

where the right-hand side comes from half the Maslov index of  $F$ , suitably extended across the boundary puncture.  $\square$

Next we prove that  $\mathcal{N}$  is nonempty for any choice of  $\eta_\varepsilon$ .

**Lemma 7.13.8.** *Let  $u_0 = \mathfrak{E}^{-1}(1, 0)$ . Then  $u_0 : \overline{\mathbb{D}} \rightarrow \mathbb{C}$  is a meromorphic function with a pole at  $-1$  and  $u_0(e^{i\theta}) \in \mathbb{R}^+ \cdot e^{i\eta_\varepsilon(\theta)}$ .*

*Proof.* The only nontrivial part of the statement is that  $u_0$  has no zeros along the boundary. In this case the analog of Equation (7.13.1) is:

$$\sum_{z \in \text{int}(\Sigma)} \nu(z) + \frac{1}{2} \sum_{z \in \partial \Sigma} \nu(z) = 0,$$

which implies that  $u_0$  has no zeros.  $\square$

**Corollary 7.13.9.**  *$\mathcal{N}$  is nonempty for any choice of  $\eta_\varepsilon$ .*

*Proof.* Take any  $u \in \tilde{\mathcal{N}}$  with  $c_2 \neq 0$ . Then  $u + cu_0 \in \mathcal{N}$  for any  $c$  sufficiently large.  $\square$

**Lemma 7.13.10.** *If  $c_1, c_2 \neq 0$ , then  $\mathfrak{E}^{-1}(c_1, c_2)$  either has a simple zero in the interior or has two zeros (counted with multiplicity) along the boundary.*

*Proof.* The analog of Equation (7.13.1) for  $w = \mathfrak{E}^{-1}(c_1, c_2)$  with  $c_1, c_2 \neq 0$  is:

$$\sum_{z \in \text{int}(\Sigma)} \nu(z) + \frac{1}{2} \sum_{z \in \partial \Sigma} \nu(z) = 1,$$

due to the contribution of the pole at 0.  $\square$

In particular the maps in  $\mathcal{N}$  have a unique zero in the interior. We denote  $\mathbb{P}\mathcal{N} = \mathcal{N}/\mathbb{R}^+$ , where  $\mathbb{R}^+$  acts on  $\mathcal{N}$  by multiplication and we define the maps

$$\widehat{\mathfrak{F}} : \mathcal{N} \rightarrow \mathbb{D} - \{0\}, \quad \mathfrak{F} : \mathbb{P}\mathcal{N} \rightarrow \mathbb{D} - \{0\}$$

by  $\widehat{\mathfrak{F}}(w) = w^{-1}(0)$  and  $\mathfrak{F}([w]) = \widehat{\mathfrak{F}}(w)$ .

**Lemma 7.13.11.** *The map  $\mathfrak{F} : \mathbb{P}\mathcal{N} \rightarrow \mathbb{D} - \{0\}$  is a diffeomorphism.*

*Proof.* We first prove injectivity. Let  $w_0$  and  $w_1$  be maps in  $\mathcal{N}$  such that  $w_0^{-1}(0) = w_1^{-1}(0)$ . Then  $\omega = \frac{w_0}{w_1}$  is a holomorphic map on  $\overline{\mathbb{D}}$  such that  $\omega(\partial \overline{\mathbb{D}}) \subset \mathbb{R}$ , so it is constant. Then  $w_0$  and  $w_1$  represent the same element in  $\mathbb{P}\mathcal{N}$ .

Next we prove surjectivity. Fix an element  $w_0 \in \mathcal{N}$  and let  $z_0 \in \mathbb{D}$  such that  $w_0(z_0) = 0$ . For any  $z \in \mathbb{D} - \{0, z_0\}$  we look for a holomorphic function  $\omega_z : \overline{\mathbb{D}} \rightarrow \mathbb{CP}^1$  such that  $\omega_z(z) = 0$ ,  $\omega_z(z_0) = \infty$ , and  $\omega_z(\partial \overline{\mathbb{D}}) \subset \mathbb{R}^+$ . By the argument of the previous paragraph, if such  $\omega_z$  exists, it is unique up to multiplication by a positive real constant. Moreover,  $w = \omega_z w_0 \in \mathcal{N}$  and  $w(z) = 0$ .

We can construct  $\omega_z$  with the desired properties as

$$\omega_z(\zeta) = f((g(\zeta))^2),$$

where  $g : \overline{\mathbb{D}} \xrightarrow{\sim} \overline{\mathbb{H}}$  is a fractional linear transformation such that  $g(z_0) = i$ ,  $\text{Re}(g(z)) = 0$ , and  $0 < \text{Im}(g(z)) < 1$ , and  $f : \mathbb{CP}^1 \xrightarrow{\sim} \mathbb{CP}^1$  is an element of  $PSL(2, \mathbb{R})$  (i.e.,  $f$  fixes the real axis) such that  $f((g(z))^2) = 0$ ,  $f((g(z_0))^2) = f(-1) = \infty$ , and  $f(0) = 1$ . Note that  $f$  and  $g$  are uniquely determined by the above conditions.

Finally we prove that  $\mathfrak{F}$  is a diffeomorphism. We define  $\Xi(z) = \omega_z w_0 \in \mathcal{N}$ ; then  $\mathfrak{F}^{-1}(z)$  is the class of  $\Xi(z)$  in  $\mathbb{P}\mathcal{N}$ . The map  $\Xi$  is smooth because the maps  $f$ ,  $g$ , and hence  $\omega_z$  are rational maps whose coefficients depend smoothly on  $z$ . Then in order to prove that  $\mathfrak{F}$  is a diffeomorphism it suffices to prove that  $d\Xi(z)$  is injective for any  $z \in \mathbb{D} - \{0\}$ .

In order to distinguish the differential of  $\Xi$  at  $z \in \mathbb{D} - \{0\}$  from the differential of the function  $\Xi(z) : \overline{\mathbb{D}} - \{0, -1\} \rightarrow \mathbb{C}$ , we will use the notation  $d\Xi(z)$  for the former and  $\frac{\partial \Xi(z)}{\partial \zeta}$  for the latter. We define the map

$$F : \mathcal{N} \times (\mathbb{D} - \{0\}) \rightarrow \mathbb{C}, \quad F(w, z) = w(z).$$

By differentiating the identity  $F(\Xi(z), z) = 0$  we obtain

$$\frac{\partial F}{\partial w} d\Xi(z) + \frac{\partial F}{\partial z} = 0.$$

We observe that  $\frac{\partial F}{\partial z} \Big|_{(\Xi(z), z)} = \frac{\partial \Xi(z)}{\partial \zeta} \Big|_{\zeta=z}$ , which is invertible because  $z$  is a simple zero of  $\Xi(z)$ . Hence  $d\Xi(z)$  is injective.  $\square$



7.13.3. *Proof of Theorem 7.2.2.* We refer to Section 7.2 for the definition of the moduli spaces  $\mathcal{M}_1$ ,  $\mathcal{M}_0$ , and  $\mathcal{M}_{-1}$ , and of the gluing parameter space  $\mathfrak{P}$ . In order to simplify the exposition we will assume (without loss of generality) that all the multiplicities of  $\gamma'$  are 1 and that each of  $\mathcal{M}_1/\mathbb{R}$ ,  $\mathcal{M}_0$ , and  $\mathcal{M}_{-1}/\mathbb{R}$  is connected; in particular, both  $\mathcal{M}_0$  and  $\mathcal{M}_{-1}/\mathbb{R}$  are single points.

We choose a smooth slice  $\widetilde{\mathcal{M}}_1$  of the  $\mathbb{R}$ -action on  $\mathcal{M}_1$  such that the following hold for some  $\kappa'_0, \kappa_0 > 0$ ,  $K_0 > \pi - \varepsilon$ , and for all  $\bar{v}_1 \in \widetilde{\mathcal{M}}_1$ :

- each component of  $\bar{v}_1|_{s \leq 0}$  is  $(\kappa'_0 + \kappa_0, 0)$ -close to a cylinder over a component of  $\delta_0 \gamma'$ ; and
- the component  $\tilde{v}_1$  of  $\bar{v}_1|_{s \leq 0}$  which is close to  $\sigma'_\infty$  satisfies

$$(7.13.2) \quad \left| \pi \circ \tilde{v}_1 - \kappa'_0 e^{(\pi-\varepsilon)s} (ce^{\pi it}) \right| \leq \kappa_0 e^{K_0 s},$$

where  $ce^{\pi it}$  is the normalized asymptotic eigenfunction corresponding to the negative end  $\delta_0$  of  $\bar{v}_1$  and  $\pi = \pi_{D_{\rho_0}}$  is the projection to  $D_{\rho_0}$  with respect to balanced coordinates. By a slight abuse of notation, we will refer to  $\kappa'_0 e^{(\pi-\varepsilon)s} (ce^{\pi it})|_{s=0} = \kappa'_0 ce^{\pi it}$  as *the* asymptotic eigenfunction of  $\bar{v}_1$  at  $\delta_0$ .

Similarly, we choose a smooth slice  $\widetilde{\mathcal{M}}_{-1}$  of the  $\mathbb{R}$ -action on  $\mathcal{M}_{-1}$  such that the following hold for some  $\kappa'_1, \kappa_1 > 0$ ,  $K_1 > 2\varepsilon$ :

- each component of  $\bar{v}_{-1}|_{s \geq 0}$  is  $(\kappa'_1 + \kappa_1, 0)$ -close to a strip over a component of  $\{z_\infty\} \cup \mathbf{y}'$ ; and
- the component  $\tilde{v}_{-1}$  of  $\bar{v}_{-1}|_{s \geq 0}$  which is close to  $\sigma_\infty$  satisfies

$$(7.13.3) \quad \left| \pi \circ \tilde{v}_{-1} - \kappa'_1 e^{-2\varepsilon s} (de^{-\varepsilon it}) \right| \leq \kappa_1 e^{-K_1 s},$$

where  $de^{-\varepsilon it}$  is a normalized asymptotic eigenfunction corresponding to the positive end  $z_\infty$  of  $\bar{v}_{-1}$ . Note that  $d$  is completely determined by the data  $\{(i, j) \rightarrow (i, j)\}$  at the positive end  $z_\infty$ . Without loss of generality, we assume that  $\bar{a}_{i,j} = \mathbb{R}^+$ ,  $\bar{h}(\bar{a}_{i,j}) = \mathbb{R}^+ \cdot e^{i\varepsilon}$ , so that  $d = e^{i\varepsilon}$ . Similarly, we refer to  $\kappa'_1 e^{-2\varepsilon s} de^{-\varepsilon it}|_{s=0} = \kappa'_1 de^{-\varepsilon it}$  as *the* asymptotic eigenfunction of  $\bar{v}_{-1}$  at  $z_\infty$ .

In the rest of this section, when we write  $\bar{v}_i$ ,  $i = 1, -1$ , we will assume that  $\bar{v}_i$  is in the slice  $\widetilde{\mathcal{M}}_i$ . Also, for  $T \in \mathbb{R}$  and  $i = \pm 1$ , let  $\bar{v}_{i,T}$  be the  $T$ -translates of  $\bar{v}_i$  in the  $\mathbb{R}$ -direction; i.e., if  $s : \overline{W}^* \rightarrow \mathbb{R}$ ,  $*$  =  $\emptyset, '$ , is the  $\mathbb{R}$ -coordinate, then  $s \circ \bar{v}_{i,T} = s \circ \bar{v}_i + T$ . Let  $f(t) = \beta e^{\pi it}$  be the asymptotic eigenfunction of  $\bar{v}_1$  at  $\delta_0$  with eigenvalue  $\pi - \varepsilon$ . Then the asymptotic eigenfunction of  $\bar{v}_{1,T}$  at  $\delta_0$  is  $e^{-(\pi-\varepsilon)T} f$ . Similarly, let  $g(t) = \alpha_0 e^{i\varepsilon(1-t)}$  be the asymptotic eigenfunction of  $\bar{v}_{-1}$  at  $z_\infty$  with eigenvalue  $2\varepsilon$ . Then the asymptotic eigenfunction of  $\bar{v}_{-1,T}$  at  $z_\infty$  is  $e^{2\varepsilon T} g$ .

In this subsection we fix  $\gamma \in \widetilde{\mathcal{O}}_{2g}$  and  $\mathbf{y} \in \mathcal{S}_{\mathbf{a}, h(\mathbf{a})}$  once and for all. Since there is no risk of confusion, we will write

$$\mathcal{M} = \mathcal{M}_{\mathcal{J}_-}^{I=3, n^-=m}(\gamma, \mathbf{y}), \quad \mathcal{M}^{ext} = \mathcal{M}_{\mathcal{J}_-}^{I=3, n^-=m, ext}(\gamma, \mathbf{y}).$$

After identifying the quotient  $\mathcal{M}_i/\mathbb{R}$  with the slice  $\widetilde{\mathcal{M}}_i$  for  $i = \pm 1$  we write

$$\mathfrak{P} = (5r, \infty)^2 \times \widetilde{\mathcal{M}}_1 \times \mathcal{M}_0 \times \widetilde{\mathcal{M}}_{-1};$$

compare with Equation (7.2.2). We describe the gluing map

$$G : \mathfrak{P} \rightarrow \mathcal{M}^{ext}, \quad \mathfrak{d} = (T_{\pm}, \bar{v}_1, \bar{v}_0, \bar{v}_{-1}) \mapsto \bar{u}(\mathfrak{d}),$$

defined in a manner similar to that of Section 6.5.2. First we form the preglued curve  $\bar{u}^{\#}(\mathfrak{d})$  from the data  $\mathfrak{d} = (T_{\pm}, \bar{v}_1, \bar{v}_0, \bar{v}_{-1})$  by patching together

$$(\bar{v}_{1,2T_+})|_{s \geq T_+}, \quad \bar{v}_0|_{-T_- \leq s \leq T_+}, \quad (\bar{v}_{-1,-2T_-})|_{s \leq -T_-}.$$

Then we choose cutoff functions  $\tilde{\beta}_i$ ,  $i = -1, 0, 1$ , on the domain of  $\bar{v}_i$ . If  $\psi_{-1}$ ,  $\psi_0$ , and  $\psi_1$  are deformations of  $\bar{v}_{-1,-2T_-}$ ,  $\bar{v}_0$ , and  $\bar{v}_{1,2T_+}$ , the condition for the deformation  $\tilde{\beta}_{-1}\psi_{-1} + \tilde{\beta}_0\psi_0 + \tilde{\beta}_1\psi_1$  of  $\bar{u}^{\#}(\mathfrak{d})$  to be holomorphic is given by the following equation:

$$(7.13.4) \quad \tilde{\beta}_{-1}\Theta_{-1}(\psi_{-1}, \psi_0) + \tilde{\beta}_0\Theta_0(\psi_{-1}, \psi_0, \psi_1) + \tilde{\beta}_1\Theta_1(\psi_0, \psi_1) = 0,$$

Equation (7.13.4) is analogous to Equation (6.5.1) and is solved as in Step 3 of Section 6.5.2. The situation considered here is simpler than that of Section 6.5.2: in fact there is no obstruction bundle because  $\bar{v}_0$  is regular. For  $r$  sufficiently large,  $G$  is a homeomorphism onto its image. Given  $\delta > 0$  small, let  $D_{\delta}(\bar{\mathfrak{m}}^b)$  be the closed ball of radius  $\delta$  and center  $\bar{\mathfrak{m}}^b$  in  $B_-$  with respect to some fixed metric. Then let  $\mathcal{M}_{(\kappa, \nu)}^{\delta} \subset \mathcal{M}^{ext}$  be the subset of curves  $\bar{u}$  that pass through  $D_{\delta}(\bar{\mathfrak{m}}^b) \times \{z_{\infty}\} \subset \bar{W}_-$  and are  $(\kappa, \nu)$ -close to breaking.

**Claim 7.13.12.**  $\mathcal{M}_{(\kappa, \nu)}^{\delta} \subset \mathcal{M}$ .

*Proof.* Something stronger is actually true: if the image of  $\bar{u} \in \mathcal{M}^{ext}$  intersects the section at infinity  $\sigma_{\infty}^-$  at a point in the interior of  $\sigma_{\infty}^-$ , then  $\bar{u} \in \mathcal{M}$ . This is a consequence of the intersection theory developed in Section 7.5: in fact  $n^-(\bar{u}) = m$  and the intersection point with  $\sigma_{\infty}^-$  at the interior already contributes  $m$  to  $n^-(\bar{u})$ . However, if  $\bar{u} \in \mathcal{M}^{ext} - \mathcal{M}$ , then the image of  $\bar{u}$  also intersects  $\sigma_{\infty}^-$  at some boundary points, which give some extra contribution to  $n^-(\bar{u})$ . This is a contradiction.  $\square$

Let  $\mathfrak{d} = (T_{\pm}, \bar{v}_1, \bar{v}_0, \bar{v}_{-1}) \in \mathfrak{P}$ . Writing  $\bar{u}^{\#} = \bar{u}^{\#}(\mathfrak{d})$  and  $\bar{u} = \bar{u}(\mathfrak{d})$ , we define  $w^{\#} = w^{\#}(\mathfrak{d})$  and  $w = w(\mathfrak{d})$  as follows:

$$(7.13.5) \quad w^{\#} = e^{\varepsilon s} \cdot (\pi \circ \bar{u}^{\#}|_{-2T_- \leq s \leq 2T_+}) \quad \text{and} \quad w = e^{\varepsilon s} \cdot (\pi \circ \bar{u}|_{-2T_- \leq s \leq 2T_+}).$$

The map  $w$  is holomorphic by Lemma 7.9.6.

**Lemma 7.13.13.** *Let  $\mathfrak{d}_i = (T_{+,i}, T_{-,i}, \bar{v}_{1,i}, \bar{v}_0, \bar{v}_{-1}) \in \mathfrak{P}$  be a sequence such that  $\lim_{i \rightarrow \infty} T_{\pm,i} = +\infty$  and  $\lim_{i \rightarrow \infty} \bar{v}_{1,i} = \bar{v}_{1,\infty} \in \widetilde{\mathcal{M}}_1$ . Then the sequence  $\bar{u}_i = G(\mathfrak{d}_i)$  has a subsequence which converges to  $(\bar{v}_{1,\infty}, \bar{v}_0, \bar{v}_{-1})$ .*

*Proof.* The lemma is a consequence of the following standard gluing result (cf. [HT2, Theorem 7.3(a)]): Given  $(\kappa, \nu)$ , there exists  $r \gg 0$  such that  $G(\mathfrak{d})$  is  $(\kappa, \nu)$ -close to  $\mathfrak{d}$  for each  $\mathfrak{d} \in \mathfrak{P} = (5r, \infty)^2 \times \widetilde{\mathcal{M}}_1 \times \mathcal{M}_0 \times \widetilde{\mathcal{M}}_{-1}$ .  $\square$

**Lemma 7.13.14.** *Let  $\mathfrak{d}_n = (T_{+,n}, T_{-,n}, \bar{v}_{1,n}, \bar{v}_0, \bar{v}_{-1})$  be a sequence such that  $\lim_{n \rightarrow +\infty} T_{\pm,n} = +\infty$ . Let  $\bar{\mathfrak{m}}_n = \bar{u}(\mathfrak{d}_n)^{-1}(\sigma_{\infty}^-)$  and let  $f_n(t) = \beta_n e^{\pi i t}$  be the asymptotic eigenfunction of  $\bar{v}_{1,n}$  at the negative end  $\delta_0$ . Then, after rescaling by positive*

real numbers and taking a subsequence, the sequence  $w_n = w(\mathfrak{d}_n)$  of holomorphic functions defined as in Equation (7.13.5) converges in the  $C_{loc}^\infty$ -topology to a holomorphic map  $w_\infty \in \tilde{\mathcal{N}}$ . Moreover,  $w_\infty$  satisfies the following conditions:

- (1) if  $\lim_{n \rightarrow +\infty} \beta_n = \beta_\infty$ , then:
  - (i)  $\mathfrak{E}(w_\infty) = (\lambda_1 \alpha_0, \lambda_2 \beta_\infty)$  for some  $\lambda_1, \lambda_2 \geq 0$ ,  $\lambda_1, \lambda_2$  not both zero;
  - (ii)  $\Pi(\mathfrak{d}_n)$ , after rescaling by positive constants, limits to  $\mathfrak{E}(w_\infty)$ ;
- (2) if  $\lim_{n \rightarrow +\infty} \bar{\mathfrak{m}}_n = \bar{\mathfrak{m}}_\infty \in \text{int}(B_-)$ , then  $w_\infty(\bar{\mathfrak{m}}_\infty) = 0$  and  $w_\infty \in \mathcal{N}$ .

*Proof.* By Lemma 7.13.13 we can extract a subsequence (which we still call  $\mathfrak{d}_i$ ) such that  $\bar{u}(\mathfrak{d}_i)$  converges to  $\bar{u}_\infty = (\bar{v}_{1,\infty}, \bar{v}_0, \bar{v}_{-1})$ . By the SFT convergence of  $\bar{u}(\mathfrak{d}_i)$  to  $\bar{u}_\infty$  we obtain a sequence of good truncations satisfying the estimates of Lemma 7.9.4. The proof of Theorem 7.9.15 then goes through essentially unchanged.  $\square$

Let  $w_0 \in \mathcal{N}$  be a map such that  $w_0(\bar{\mathfrak{m}}^b) = 0$  and let  $\mathcal{U}_\delta \subset \mathcal{N}$  be the subset consisting of maps  $w$  such that  $w(\bar{\mathfrak{m}}) = 0$  for some  $\bar{\mathfrak{m}} \in D_\delta(\bar{\mathfrak{m}}^b)$ . Note that  $\mathcal{U}_\delta$  is an  $\mathbb{R}^+$ -invariant open neighborhood of  $w_0$ . We now have the following diagram:

$$\begin{array}{ccc} \mathfrak{P} & \xrightarrow{G} & \mathcal{M}^{ext} \supset \mathcal{M}_{(\kappa, \nu)}^\delta \\ \Pi \downarrow & & \\ \mathbb{R}^+ \times \mathbb{C}^\times & \xleftarrow{\mathfrak{E}} & \mathcal{N} \supset \mathcal{U}_\delta. \end{array}$$

The map  $\Pi$  is defined as follows:

$$\Pi(T_\pm, \bar{v}_1, \bar{v}_0, \bar{v}_{-1}) = (\alpha, \beta),$$

where  $\alpha e^{i\varepsilon(1-t)}$  is the asymptotic eigenfunction of  $\bar{v}_{-1, -2T_-}$  at  $z_\infty$  and  $\beta e^{i\pi t}$  is the asymptotic eigenfunction of  $\bar{v}_{1, 2T_+}$  at  $\delta_0$ . Let us write  $\mathfrak{P}_\delta := \Pi^{-1}(\mathfrak{E}(\mathcal{U}_\delta))$ . Then  $\mathfrak{P}_\delta \subset \mathfrak{P}$  is an open set since  $\mathfrak{E}$  is a diffeomorphism.

**Lemma 7.13.15.** *If  $r$  is sufficiently large, then the following hold:*

- (1)  $\mathcal{M}_{(\kappa, \nu)}^{\delta/3} \cap \text{Im}(G) \subset G(\mathfrak{P}_{\delta/2})$ .
- (2)  $G(\mathfrak{P}_{2\delta/3}) \subset \text{int}(\mathcal{M}_{(\kappa, \nu)}^\delta)$ .

*Proof.* (1) We show that, for  $r$  sufficiently large, if

$$\bar{u}(\mathfrak{d})^{-1}(\sigma_\infty^-) \in D_{\delta/3}(\bar{\mathfrak{m}}^b),$$

then  $\mathfrak{d} \in \mathfrak{P}_{\delta/2}$ . Arguing by contradiction, suppose there exist sequences  $r_i \rightarrow \infty$ ,  $\mathfrak{d}_i = (T_{\pm, i}, \bar{v}_{1, i}, \bar{v}_0, \bar{v}_{-1}) \notin \mathfrak{P}_{\delta/2}$ , and  $\bar{\mathfrak{m}}_i \in D_{\delta/3}(\bar{\mathfrak{m}}^b)$  such that  $T_{\pm, i} > 5r_i$  and  $\bar{u}(\mathfrak{d}_i)^{-1}(\sigma_\infty^-) = \bar{\mathfrak{m}}_i$ . By passing to a subsequence, we may assume that  $\lim_{i \rightarrow \infty} \bar{\mathfrak{m}}_i = \bar{\mathfrak{m}}_\infty \in D_{5\delta/12}(\bar{\mathfrak{m}}^b)$  and that the asymptotic eigenfunctions of  $\bar{v}_{1, i}$  converge to  $\beta_\infty e^{\pi i t}$  for  $i \rightarrow \infty$ . We apply Lemma 7.13.14 to  $\bar{u}(\mathfrak{d}_i)$  to obtain a holomorphic map  $w_\infty \in \mathcal{N}$  such that  $w_\infty(\bar{\mathfrak{m}}_\infty) = 0$  and Lemma 7.13.14(i) and (ii) hold. Since  $\mathfrak{d}_i \notin \mathfrak{P}_{\delta/2}$  for all  $i$ , (ii) implies that  $(\lambda_1 \alpha_0, \lambda_2 \beta_\infty) \notin \mathfrak{E}(\mathcal{U}_{\delta/2})$ . On the other hand,  $w_\infty$  has a zero in  $D_{5\delta/12}(\bar{\mathfrak{m}}^b)$ , a contradiction.

(2) is similar to (1). Suppose there exist sequences  $r_i \rightarrow \infty$  and  $\mathfrak{d}_i = (T_{\pm,i}, \bar{v}_{1,i}, \bar{v}_0, \bar{v}_{-1}) \in \mathfrak{P}_{2\delta/3}$  such that  $T_{\pm,i} > 5r_i$  and  $\bar{u}(\mathfrak{d}_i) \notin \text{int}(\mathcal{M}_{(\kappa,\nu)}^\delta)$ . By passing to a subsequence and applying Lemma 7.13.14 to  $\bar{u}(\mathfrak{d}_i)$ , we obtain a holomorphic map  $w_\infty \in \tilde{\mathcal{N}}$  such that  $\mathfrak{E}(w_\infty) \in \mathfrak{E}(\mathcal{U}_{2\delta/3})$  by Lemma 7.13.14(ii). On the other hand,  $w_\infty^{-1}(0) \notin D_\delta(\bar{\mathfrak{m}}^b)$  since  $\bar{u}(\mathfrak{d}_i)^{-1}(\sigma_\infty^-) \notin D_\delta(\bar{\mathfrak{m}}^b)$  by Lemma 7.13.14(b). This is a contradiction.  $\square$

Fix  $\delta > 0$  and take  $r_0 > 5r$  sufficiently large. We write

$$\partial\mathfrak{P}_{2\delta/3,(r_0)} := \mathfrak{P}_{2\delta/3} \cap \partial\mathfrak{P}_{(r_0)},$$

where  $\mathfrak{P}_{(r_0)} = \{T_+ \geq r_0\} \subset \mathfrak{P}$  and  $\partial\mathfrak{P}_{(r_0)} = \{T_+ = r_0\}$ . By Lemma 7.13.15(2), we can define

$$(7.13.6) \quad \Upsilon' : \partial\mathfrak{P}_{2\delta/3,(r_0)} \rightarrow D_\delta(\bar{\mathfrak{m}}^b),$$

which maps  $\mathfrak{d}$  to  $G(\mathfrak{d})^{-1}(\sigma_\infty^-)$ . Since  $\partial\mathfrak{P}_{2\delta/3,(r_0)}$  is compact, the map  $\Upsilon'$  is proper and the local degree is well-defined.

Also define

$$(7.13.7) \quad \Upsilon'' = \widehat{\mathfrak{F}} \circ \mathfrak{E}^{-1} \circ \Pi : \partial\mathfrak{P}_{2\delta/3,(r_0)} \rightarrow D_\delta(\bar{\mathfrak{m}}^b).$$

The map  $\Upsilon''$  is well-defined by the definitions of  $\mathfrak{P}_{2\delta/3}$  and  $\mathcal{U}_\delta$ . It is also proper and the local degree is well-defined.

The maps  $\Upsilon'$  and  $\Upsilon''$  are sufficiently close in the following sense.

**Lemma 7.13.16.** *For any  $k$ ,  $\Upsilon'$  and  $\Upsilon''$  can be made arbitrarily  $C^0$ -close by choosing  $r_0$  sufficiently large.*

*Proof.* Follows from Lemma 7.13.14.  $\square$

In particular, the local degrees of  $\Upsilon'$  and  $\Upsilon''$  near  $\bar{\mathfrak{m}}^b$  agree. The local degree of  $\Upsilon'$  is equal to the left-hand side of Equation (7.2.3), while the local degree of  $\Upsilon''$  is equal to the right-hand side of Equation (7.2.3). (We were assuming that each of  $\mathcal{M}_1/\mathbb{R}$ ,  $\mathcal{M}_0$ , and  $\mathcal{M}_{-1}/\mathbb{R}$  is connected.) This completes the proof of Theorem 7.2.2.

*Acknowledgements.* We are indebted to Michael Hutchings for many helpful conversations and for our previous collaboration which was a catalyst for the present work. We also thank Denis Auroux, Tobias Ekholm, Dusa McDuff, Ivan Smith and Jean-Yves Welschinger for illuminating exchanges. Part of this work was done while KH and PG visited MSRI during the academic year 2009–2010. We are extremely grateful to MSRI and the organizers of the “Symplectic and Contact Geometry and Topology” and the “Homology Theories of Knots and Links” programs for their hospitality; this work probably would never have seen the light of day without the large amount of free time which was made possible by the visit to MSRI. KH also thanks the Simons Center for Geometry and Physics for their hospitality during his visit in May 2011.

## REFERENCES

- [Ab] C. Abbas, *Finite energy surfaces and the chord problem*, Duke Math. J. **96** (1999), 241–316.
- [Bo1] F. Bourgeois, *A Morse Bott approach to contact homology*, from: “Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montréal, QC, 2001)”, Fields Inst. Comm. **35**, Amer. Math. Soc., Providence, RI (2003) 55–77.
- [Bo2] F. Bourgeois, *A Morse-Bott approach to contact homology*, Ph.D. thesis, 2002.
- [BEE] F. Bourgeois, T. Ekholm and Y. Eliashberg, *Effect of Legendrian surgery. With an appendix by Sheel Ganatra and Maksim Maydanskiy*, Geom. Topol. **16** (2012), 301–389.
- [BEHWZ] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. **7** (2003), 799–888 (electronic).
- [CGH0] V. Colin, P. Ghiggini and K. Honda, *Equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions*, Proc. Nat. Acad. Sci. **108** (2011), 8101–8105.
- [CGH1] V. Colin, P. Ghiggini and K. Honda, *Embedded contact homology and open book decompositions*, preprint 2010. [ArXiv:1008.2734](#).
- [CGH2] V. Colin, P. Ghiggini and K. Honda, *The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions II*, in preparation.
- [CGH3] V. Colin, P. Ghiggini and K. Honda, *The equivalence of Heegaard Floer homology and embedded contact homology III: from hat to plus*, in preparation.
- [CGHH] V. Colin, P. Ghiggini, K. Honda and M. Hutchings, *Sutures and contact homology I*, Geom. Topol. **15** (2011), 1749–1842.
- [CHL] V. Colin, K. Honda and F. Laudenbach, *On the flux of pseudo-Anosov homeomorphisms*, Algebr. Geom. Topol. **8** (2008), 2147–2160.
- [DS] S. Donaldson and I. Smith, *Lefschetz pencils and the canonical class for symplectic four-manifolds*, Topology **42** (2003), 743–785.
- [Dr] D. Dragnev, *Fredholm theory and transversality for noncompact pseudoholomorphic maps in symplectizations*, Comm. Pure Appl. Math. **57** (2004), 726–763.
- [EGH] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory*, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. 2000, Special Volume, Part II, 560–673.
- [Gi1] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. **66** (1991), 637–677.
- [Gi2] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414, Higher Ed. Press, Beijing, 2002.
- [Gr] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.
- [Ho1] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. **114** (1993), 515–563.
- [Ho2] H. Hofer, *Holomorphic curves and dynamics in dimension three*, Symplectic geometry and topology (Park City, UT, 1997), 35–101, IAS/Park City Math. Ser., 7, Amer. Math. Soc., Providence, RI, 1999.
- [HLS] H. Hofer, V. Lizan and J.-C. Sikorav, *On genericity for holomorphic curves in four-dimensional almost complex manifolds*, J. Geom. Anal. **7** (1997), 149–159.
- [HWZ1] H. Hofer, K. Wysocki and E. Zehnder, *Properties of pseudoholomorphic curves in symplectisations. I. Asymptotics*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 337–379.
- [HWZ2] H. Hofer, K. Wysocki and E. Zehnder, *Properties of pseudo-holomorphic curves in symplectizations. II. Embedding controls and algebraic invariants*, Geom. Funct. Anal. **5** (1995), 270–328.
- [HKM1] K. Honda, W. Kazez and G. Matić, *On the contact class in Heegaard Floer homology*, J. Differential Geom. **83** (2009), 289–311.
- [HKM2] K. Honda, W. Kazez and G. Matić, *Contact structures, sutured Floer homology, and TQFT*, preprint 2008. [ArXiv:0807.2431](#).
- [Hu1] M. Hutchings, *An index inequality for embedded pseudoholomorphic curves in symplectizations*, J. Eur. Math. Soc. (JEMS) **4** (2002), 313–361.

- [Hu2] M. Hutchings, *The embedded contact homology index revisited*, New perspectives and challenges in symplectic field theory, 263–297, CRM Proc. Lecture Notes, 49, AMS, 2009.
- [HS1] M. Hutchings and M. Sullivan, *The periodic Floer homology of a Dehn twist*, Algebr. Geom. Topol. **5** (2005), 301–354.
- [HS2] M. Hutchings and M. Sullivan, *Rounding corners of polygons and the embedded contact homology of  $T^3$* , Geom. Topol. **10** (2006), 169–266 (electronic).
- [HT1] M. Hutchings and C. Taubes, *Gluing pseudoholomorphic curves along branched covered cylinders I*, J. Symplectic Geom. **5** (2007), 43–137.
- [HT2] M. Hutchings and C. Taubes, *Gluing pseudoholomorphic curves along branched covered cylinders II*, J. Symplectic Geom. **7** (2009), 29–133.
- [HT3] M. Hutchings and C. Taubes, *Proof of the Arnold chord conjecture in three dimensions I*, Math. Res. Lett. **18** (2011), 295–313.
- [IP1] E. Ionel and T. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. (2) **157** (2003), 45–96.
- [IP2] E. Ionel and T. Parker, *The symplectic sum formula for Gromov-Witten invariants*, Ann. of Math. (2) **159** (2004), 935–1025.
- [KM] M. Kontsevich and Y. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Commun. Math. Phys. **164** (1994), 525–562.
- [KrM] P. Kronheimer and T. Mrowka, *Monopoles and three-manifolds*, New Mathematical Monographs, 10. Cambridge University Press, Cambridge, 2007.
- [KLT1] C. Kutluhan, Y. Lee and C. Taubes, *HF=HM I: Heegaard Floer homology and Seiberg-Witten Floer homology*, preprint 2010. [ArXiv:1007.1979](#).
- [KLT2] C. Kutluhan, Y. Lee and C. Taubes, *HF=HM II: Reeb orbits and holomorphic curves for the ech/Heegaard-Floer correspondence*, preprint 2010. [ArXiv:1008.1595](#).
- [KLT3] C. Kutluhan, Y. Lee and C. Taubes, *HF=HM III: Holomorphic curves and the differential for the ech/Heegaard Floer correspondence*, preprint 2010. [ArXiv:1010.3456](#).
- [KLT4] C. Kutluhan, Y. Lee and C. Taubes, *HF=HM IV: The Seiberg-Witten Floer homology and ech correspondence*, preprint 2011. [ArXiv:1107.2297](#).
- [KLT5] C. Kutluhan, Y. Lee and C. Taubes, *HF=HM V: Seiberg-Witten-Floer homology and handle addition*, preprint 2012. [ArXiv:1204.0115](#).
- [LT] Y. Lee and C. Taubes, *Periodic Floer homology and Seiberg-Witten Floer cohomology*, preprint 2009. [ArXiv:0906.0383](#).
- [LR] A. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. **145** (2001), 151–218.
- [Li] R. Lipshitz, *A cylindrical reformulation of Heegaard Floer homology*, Geom. Topol. **10** (2006), 955–1097.
- [M1] D. McDuff, *Singularities and positivity of intersections of  $J$ -holomorphic curves*. With an appendix by Gang Liu. Progr. Math., 117, Holomorphic curves in symplectic geometry, 191–215, Birkhäuser, Basel, 1994.
- [M2] D. McDuff, *Comparing absolute and relative Gromov-Witten invariants*, preprint 2008.
- [MS] D. McDuff and D. Salamon,  *$J$ -holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, 52. American Mathematical Society, Providence, RI, 2004.
- [MW] M. Micallef and B. White, *The structure of branch points in minimal surfaces and in pseudoholomorphic curves*, Ann. of Math. (2) **141** (1995), 35–85.
- [OSz1] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), 1027–1158.
- [OSz2] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) **159** (2004), 1159–1245.
- [Ra] J. Rasmussen, *Floer homology and knot complements*, Ph.D. thesis, 2003. [ArXiv:math.GT/0306378](#).
- [Se1] P. Seidel, *A long exact sequence for symplectic Floer cohomology*, Topology **42** (2003), 1003–1063.

- [Se2] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*. Zürich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.
- [T1] C. Taubes, *The Seiberg-Witten equations and the Weinstein conjecture*, Geom. Topol. **11** (2007), 2117–2202.
- [T2] C. Taubes, *Embedded contact homology and Seiberg-Witten Floer cohomology I–V*, Geom. Topol. **14** (2010) 2497–3000.
- [T3] C. Taubes, *The structure of pseudoholomorphic subvarieties for a degenerate almost complex structure and symplectic form on  $S^1 \times B^3$* , Geom. Topol. **2** (1998), 221–332.
- [U] I. Ustilovsky, *Infinitely many contact structures on  $S^{4m+1}$* , Internat. Math. Res. Notices **14** (1999), 781–791.
- [We1] C. Wendl, *Finite energy foliations on overtwisted contact manifolds*, Geom. Topol. **12** (2008) 531–616.
- [We2] C. Wendl, *Finite energy foliations and surgery on transverse links*, Ph.D. thesis, 2005.
- [We3] C. Wendl, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*, preprint 2008. ArXiv:0802.3842.
- [We4] C. Wendl, *Punctured holomorphic curves with boundary on 3-manifolds: Fredholm theory and embeddedness*, in preparation.

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